PART A:

1. a b d
2. a d
3. a b d
4. c
5. a b d
6. a d
7. a b c d
8. a c
9. a b c
10. b

PART B:

1. Since there is restriction on the girls seating, let us seat the 8 boys first in 8! ways. There are 9 gaps (including the two ends). We can select 7 of these gaps and seat the girls. That is

$$\binom{9}{7}7!$$

ways. So the answer is the product of these two numbers, that is,

$$\binom{9}{7}7!8!$$

Note that any arrangement in the required fashion can be achieved by this procedure.

2. For any positive number $x$ we have $\log(1+x) \leq x$. This is because $\log(1+x) - x$ equals zero for $x = 0$ and its derivative equals

$$\frac{1}{1+x} - 1 \leq 0$$

so that the function is decreasing. Thus $\log(1+x)-x \leq 0$ when $x \geq 0$.

$$\log(n+1) - \log n = \log(1+\frac{1}{n}) \leq \frac{1}{n}.$$

Hence $n$-th term of the series is at most

$$\frac{1}{n\sqrt{n}}.$$

Since the series

$$\frac{1}{n^p}$$

converges for $p > 1$ and the given series of positive numbers is dominated by this series, the given series converges.

3. Consider calculating the Riemann integral

$$\int_0^t \sin x dx.$$

If we take the the partition

$$0 < \frac{t}{n} < \frac{2t}{n} < \frac{3t}{n} < \cdots < \frac{nt}{n} = t$$

and take the left end point in each interval to calculate the Riemann sum we get

$$\frac{t}{n}\{\sin 0 + \sin\frac{t}{n} + \sin\frac{2t}{n} + \cdots + \sin\frac{(n-1)t}{n}\}.$$

Since the Riemann sum converges to the integral we conclude that the limit of the above sums equals the integral. Since the last two terms converge to zero as $n \to \infty$, we get

$$\frac{t}{n}\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \cdots + \sin\frac{(n-3)t}{n}\} \to \int_0^t \sin x dx = 1 - \cos t.$$

After dividing by $t$ we get the given assertion.

4.
$$\left|\frac{\sin(n^3 x)}{n^2+x^2}\right| \leq \frac{1}{n^2}$$

so the integral (in modulus) is at most $2\pi n/n^2 = 2\pi/n \to 0$.

5. The $n$-th term of the series is dominated (in modulus) by $1/n^2$ and the series $\sum \frac{1}{n^2}$ is convergent. By Weierstrass $M$-test the given series of functions converges uniformly over $R$.

6. If $f(x_k)$ is the largest among the $n$ values of $f$, namely, $f(x_1), f(x_2), \cdots, f(x_n)$ and if $f(x_l)$ is the minimum, then the average lies between these two. The intermediate value theorem for continuous functions shows that there is a point $u$ between $x_k$ and $x_l$ such that $f(u)$ equals the average.

7. Let, if possible $f$ be such a function. Let $a < b$ be the two points with $f(a) = f(b) = 0$.Thus $f$ does not assume the value zero in the three intervals $(-\infty, a)$; $(a, b)$, $(b, \infty)$ and hence in each of the intervals it keeps the same sign.

   Assume without loss of generality $f$ is strictly positive in $(a, b)$ [otherwise we repeat similar argument or we use the function $-f$]. Let $M$ be the maximum of $f$ in the compact interval $[a, b]$. Since $f$ is not zero in the open interval, we observe that $M > 0$. Let $x_0 \in (a, b)$ be such tat $f(x_0) = M$ There can not be two such points $x_0 < x_1$. Indeed, if there are two such points, then $f$ takes all values between zero and $M$ in the interval $[a, x_0]$ and also in the interval $[x_1, b]$. Thus in $(x_0, x_1)$ it assumes some of the values for a third time.

   Thus there is only one such $x_0$ and all values between zero and $M$ are assumed by $f$ in $(a, x_0)$ once and in $(x_0, b)$ once. But the value $M$ itself must be assumed again for a second time, say it happens at $x_1 \in (b, \infty)$. Since $f(b) = 0$ we conclude that $f$ assumes all values between zero and $M$ again for a third time during $(b, x_1)$. This contradiction proves the result.

8. Since $|x^2 - y^2| \leq x^2 + y^2$ we get,

$$0 \leq |f(x, y)| \leq |x|.$$

   If $(x, y) \to (0, 0)$, we have $x \to 0$, So by sandwich theorem we get $\lim f(x, y) = 0$ as $(x, y) \to (0, 0)$.

9. If you add 1000×first row plus 100×second row plus 10×third row to the last row, we see that the last row is divisible by 11. Since the determinant is not altered by this procedure, we see that the determinant of the given matrix is divisible by 11.

10. if $e_1, e_2, e_3$ are the vectors $(1, 0, 0)$; $(1, 1, 0)$ and $(1, 1, 1)$ then the form a basis of $R^3$ and

$$(x, y, z) = ze_3 + (y - z)e_2 + (x - y)e_1$$

so that by linearity,

$$T(x, y, z) = ze_2 + (y - z)e_3 + (x - y)e_1 = (x, y, y - z).$$

Since the map $T$ takes the basis onto the basis, it is one-one. Thus Kernel$(T) = \{0\}$. Range$(T) = R^3$.

To show $T^3 = T$, it is enough to verify for the basis elements. Since $Te_1 = e_1$ we have $T^3 e_1 = e_1 = Te_1$.

$$T^3(e_2) = T^2(e_3) = Te_2.$$

Similarly $T^3 e_3 = Te_3$.

11.

$$T(c_1 + c_2 x + c_3 e^x + c_4 x e^x) = c_2 + (c_3 + c_4)e^x + c_4 x e^x.$$

Thus w.r.t. the given basis we have

$$T \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} c_2 \\ 0 \\ c_3 + c_4 \\ c_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Thus the matrix is as above.

12. Let $S$ be range $T$. Since $T \not\equiv 0$, we see $\dim(S)$ is at least one. If its dimension equals 3, then $T$ is invertible but it is not so since $T^2 \equiv 0$.

Finally, we show it can not have dimension 2. If it has dimension 2, then $T^2 \equiv 0$ tells that $T$ is zero on $S$, a two dimensional subspace. Thus Range $T$ is one dimensional (namely spanned by $Tv$ where $v$ is a vector which along with a basis of $S$ forms a basis of $R^3$). In other words $S$ has dimension one, contradiction.