**CHENNAI Mathematical Institute**

**MSc Applications of Mathematics**

*Entrance Examination, 2013*

Answer **all** 10 questions from Part A and 3 questions **each** from Parts B and C. Questions in Part A carry 4 marks each and those in Parts B and C carry 10 marks each.

## Part A

Answer all questions. Each multiple choice question has one or more (possibly all) correct answers.

To get credit, you have to tick all correct answers and not tick any incorrect answer. There is no partial credit and there are no negative marks.

1. Let $f(x) = x\cos x$ for $x \in R$. Then

   (A) there is a sequence $x_n \to -\infty$ such that $f(x_n) \to 0$.

   (B) There is a sequence $x_n \to \infty$ such that $f(x_n) \to \infty$.

   (C) there is a sequence $x_n \to \infty$ such that $f(x_n) \to -\infty$.

   (D) $f$ is a uniformly continuous function.

2. The equation $x^{10} + a_9x^9 + a_8x^8 + \cdots + a_1x - 5 = 0$ where the coefficients $a_i$ are real numbers,

   (A) must have one real root of multiplicity one.

   (B) must have at least two (distinct or repeated) real roots.

   (C) must have at least four real roots (distinct or repeated).

   (D) need not have any real root.

3. Let $g$ be a function defined on the interval $[0, 2]$ and $x \leq g(x) \leq x^2 - x + 1$ for $0 \leq x \leq 2$. Then

   (A) $g$ must necessarily be a polynomial.

   (B) $g$ must be continuous at $x = 1$.

   (C) $g$ must be continuous at $x = 0$ and $x = 2$.

   (D) $g$ must be a continuous function.

4. Let $x_n = \left(1 - \dfrac{1}{n}\right)\sin\dfrac{n\pi}{3}$ for $n \geq 1$. Denote $l = \liminf x_n$ and $s = \limsup x_n$. Then

   (A) $-\frac{\sqrt{3}}{2} \leq l < s \leq \frac{\sqrt{3}}{2}$

   (B) $-\frac{1}{2} \leq l < s < \frac{1}{2}$.

   (C) $l = -1$ and $s = +1$

   (D) $l = s = 0$.

5. $\lim_{x\to\infty} \dfrac{\sqrt{x^2+2x+4}}{2x+3}$ equals

(A) 0 (B) 1/2

(C) 4/3 (D) 1.

6. $\lim_{x\to 0} \dfrac{1+x+x^2-e^x}{2x^2}$ equals

(A) $-1/2$ (B) $-1/4$

(C) $+1/4$ (D) $+1/2$.

7. The series $\sum_{n-1}^{\infty} a_n$ where $a_n = (-1)^n n^4 e^{-n^2}$ is

(A) absolutely convergent.

(B) convergent but not absolutely convergent.

(C) not convergent, but partial sums oscillate between $-1$ and $+1$.

(D) partial sums are unbounded.

8. Let $U = \{(a,b,c)|a+2b-3c=0, 2a+5b+2c=0, 3a-b-4c=0\} \subset R^3$.
Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 3 & -1 & -4 \end{pmatrix}$. Then

(A) $dim(U)=0, rank(A)=3$ (B) $dim(U)=1, rank(A)=2$

(C) $dim(U)=1, rank(A)=1$ (D) $dim(U)=3, rank(A)=3$.

9. $U = span\{(1,1,-1),(2,3,-1),(3,1,-5)\}, V = span\{(1,1,-3),(3,-2,-8),(2,1,-3)\}$.
What is $U \cap V$?

(A) $U$ (B) $V$

(C) The $\{0\}$ subspace (D) None of the above.

10. $U = \{(x,y,z)|x=y=z\}, V = \{(x,y,z)|x=0\}$. What is $U+V$?

(A) $xy$ plane (B) $yz$ plane

(C) $zx$ plane (D) $R^3$.

## Part B

Answer any three questions. Each question carries 10 marks.

For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $f$ be a real vlaued function defined on the interval $[0, 1]$. If $|f|$ is Riemann integrable, then $f$ is also Riemann integrable.
2. If $A$ is a linear transformation of $R^n$ to $R^n$ of rank one, then there is a number $\alpha$ such that $A^2 = \alpha A$.
3. Let $f : R \to R$ be a differentiable function with $f'$ bounded. Then $f$ is a uniformly continuous function.
4. The matrix $\begin{pmatrix} \lambda & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & \lambda \end{pmatrix}$ is invertible for exactly two distinct real values of $\lambda$.

## Part C

Answer any three questions in the sheets provided. Each question carries 10 marks. State precisely any theorem that you use.

1. Let $f$ be a real valued continuous function on the interval $[0, 1]$ such that $\int_0^1 x^n f(x)dx = 0$ for $n = 0, 1, 2 \cdots$. Show that $f \equiv 0$.
2. $f$ is a real valued continuous function on the interval $[0, 2]$ which is differentiable at every point other than $x = 1$. Suppose that $\lim_{x \to 1} f'(x) = 5$. Show that $f$ is differentiable at $x = 1$.
3. Let $b$ be an $n \times 1$ (column) vector and $A$ be an $n \times n$ positive definite matrix. Define $P : R^n \to R$ by

$$P(x) = \frac{1}{2} x^t A x - x^t b.$$

   Here $t$ stands for the transpose. If $x_0$ is a vector such that $Ax_0 = b$ then show that $P(y) \geq P(x_0)$, for every $y \in R^n$.

4. $A$ is an $n \times n$ matrix with rational entries and $b$ is $n \times 1$ vector with rational entries. If there is a vector $x$ satisfying $Ax = b$, then show that there is such a vector with rational entries.