**CHENNAI Mathematical Institute**

**MSc Applications of Mathematics**

*Sample Entrance Examination, 2013*

Answer all 10 questions from Part A and 3 questions each from Parts B and C. Questions in Part A carry 4 marks and those in Parts B and C carry 10 marks each.

**Part A**

Answer all questions. Each multiple choice question has one or more (possible all) correct answers.

To get credit, you have to tick all correct answers and not tick any incorrect answer. There is no partial credit and there are no negative marks.

1. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

$$0 < a_n < c_n < b_n < 1 \text{ for all } n \geq 1$$

and suppose $a_n \to a$ and $b_n \to b$. Then we can conclude that

**(a):** $0 < a < \liminf c_n < 1$

**(b):** $a < \liminf c_n \leq \limsup c_n < b$.

**(c):** $0 \leq a \leq \liminf c_n < \limsup c_n \leq b \leq 1$.

**(d):** $\liminf(a_n + c_n) = a + \liminf c_n$.

2. Consider a power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose the radius of convergence of this power series is $R$, $0 < R < \infty$. Suppose also that $y \in \mathbb{R}$ is such that the series $\sum_{n=0}^{\infty} a_n y^n$ is convergent. Then we can conclude that

**(a):** The power series converges uniformly in $\{x \in \mathbb{R} : |x| < R\}$.

**(b):** $|y| \leq R$.

**(c):** For $\delta > 0$, the series converges uniformly in $\{x \in \mathbb{R} : |x| \leq (R - \delta)\}$.

**(d):** $f$ defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $\{x \in \mathbb{R} : |x| < R\}$ is a continuous function.

3. Let $f : [0, \infty) \mapsto [0, \infty)$ be a continuous function. Which of the following condditions ensure that $f$ is uniformly continuous on $[0, \infty)$?

**(a):** $\lim_{x\to\infty} \frac{f(x)}{x^2} = 0$.

**(b):** $\lim_{x\to\infty} \sqrt{x} f(x) = 0$.

**(c):** $\lim_{x\to\infty} f(\sin(x)) = 0$.

**(d):** $\lim_{x\to\infty} f(e^x) = 0$.

4. Let $f$ be a continuous function from $I$ to $\mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. Which of the following are true statements.

   **(a):** $f$ attains its supremum if $I$ is bounded.

   **(b):** $f$ attains its supremum if $I$ is closed.

   **(c):** $f$ attains its infimum if $I$ is aclosed bounded interval.

   **(d):** $f$ attains its infimum if $I$ is an open bounded interval.

5. Which of the following functions is continuously differentiable on its domain:

   **(a):** $f(x) = x^5$ for $x \geq 0$ and $f(x) = -x^3$ for $x < 0$

   **(b):** $g(x) = \exp(|x|)$ for $x \in \mathbb{R}$.

   **(c):** $h(x) = |x|\sin(x)$ for $x \in \mathbb{R}$.

   **(d):** $u(x) = \exp(|x|) + \exp(-|x|)$ for $x \in \mathbb{R}$.

6. Which of the following functions attains its supremum on its domain :

   **(a):** $f(x) = \frac{-1}{x(1-x)}$ for $0 < x < 1$.

   **(b):** $g(x) = -(1+x)(3+x^2)$ for $0 < x < 100$.

   **(c):** $h(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

   **(d):** $u(x) = \exp(-x)$ for $-100 \leq x < \infty$.

7. Which of the following infinite series converge absolutely?

   **(a):** $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$.

   **(b):** $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$.

   **(c):** $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\log(n)}$.

   **(d):** $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\log(n)}$.

8. Let $A$ be a $n \times m$ matrix with strictly positive entries with $m < n$ and let $C = AA^t$ (where $A^t$ is the transpose of $A$). Then we can conclude that

   **(a):** $C$ is a symmetric matrix.

   **(b):** $C$ is a strictly positive definite matrix.

**(c):** $C$ is non-singuar.

**(d):** There exists a vector $x \in \mathbb{R}^n$ such that $x^t C x = 0$.

9. Let $A$ be a non singular symmetric $n \times n$ matrix with strictly positive entries. Then we can colcude that

   **(a):** $\det(A)$ must be positive.

   **(b):** All eigenvalues of $A$ must be positive.

   **(c):** Trace of $A$ is strictly positive.

   **(d):** All entries of the matrix $A^{-1}$ are stricly positive.

10. Let $A$ be a $n \times m$ matrix with rank $r$. Consider the system of equations $Ax = b$ for an $n \times 1$ vector $b$. Then which of the following is always true

    **(a):** If $r = n$, then the equation $Ax = b$ admits a solution for all $b$.

    **(b):** If $r < n$ then the equation $Ax = b$ does not admit a solution for some $b$.

    **(c):** If $r = m$, then the equation $Ax = b$ admits a solution for all $b$.

    **(d):** If $r < m$ then the equation $Ax = b$ does not admit a solution for some $b$.

## Part B

Answer any three questions. For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $f_n$ be a sequence of continuous functions on $[0,1]$ converging to $f$ point wise, where $f$ is a continuous function. Then $f_n$ converges uniformly to $f$.

2. Let $g$ be defined by
   $$g(x) = |x|^3 \exp\{-|x|\} \;\; x \in \mathbb{R}.$$
   Then $g$ is a continuously differentiable function on $\mathbb{R}$.

3. Suppose $\lambda^2$ is an eigenvalue of $\mathbf{A}^2$. Then $\lambda$ is an eigenvalue of $\mathbf{A}$.

4. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that the radius of convergence $R$ of the power series $p(t) = \sum_{m=0}^{\infty} a_n t^n$ satisfies $R > 0$. The $a_n$ converges to 0.

## Part C

Answer any three questions. Each question carries 10 marks.State precisely any theorem that you use.

1. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be such that

$$\|\mathbf{x} + t\mathbf{y}\| \geq \|x\|, \quad \forall t \in \mathbb{R}.$$

   Show that $\mathbf{x} \cdot \mathbf{y} = 0$.

2. Let $\mathbf{A}$ be a $n \times n$ matrix and $\mathbf{y}$ be a $n \times 1$ matrix (vector) such that the equation

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

   for a $n \times 1$ matrix (vector) $\mathbf{y}$ admits no solution. Show that the rank of $\mathbf{A}$ is strictly less than $n$.

3. Let $\mathbf{A} = (a_{ij})$ be a $100 \times 100$ matrix defined by

$$a_{ij} = i^2 + j^2.$$

   Find the rank of $\mathbf{A}$.

4. For $n \geq 1$ let

$$a_n = \frac{(\log n)^4}{n^2}.$$

   Show that the series

$$\sum_{n=1}^{\infty} a_n$$

   converges.