1. Let \( \{a_n : n \geq 1\} \) be a sequence of real numbers such that the radius of convergence \( R \) of the power series \( p(t) = \sum_{m=0}^{\infty} a_n t^n \) satisfies \( R > 0 \). The sequence \( a_n \) converges to 0.

2. Let \( A \) be a symmetric \( n \times n \) matrix and suppose that \( A \) is positive definite. Then \( a_{jk} \leq \frac{1}{2}(a_{jj} + a_{kk}) \).

3. Let \( p(x) \) be a polynomial and \( a > 0 \) be such that \( p(a) > 0 \). Let \( q(x) = p(x) - p(a) \). Then \( (x - a) \) is a factor of \( q(x) \) but \( (x - a)^2 \) is not a factor of \( q(x) \).

4. Let \( f(x) = |x| \sin(x) \) and \( g(x) = |x| \cos(x) \) for \(-2\pi \leq x \leq 2\pi\). Then \( f \) and \( g \) are differentiable on \([-2\pi, 2\pi]\).

5. Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times n \) matrices that are positive definite such that \( a_{ij} < b_{ij} \forall i, j \).

Let \( C = (c_{ij}) \) be defined by \( c_{ij} = a_{ij} - b_{ij} \). Then \( C \) is also positive definite.

6. Let \( f_n \) be a sequence of continuous functions on \([0, 1]\) converging to \( f \) point wise, where \( f \) is a continuous function. Then \( f_n \) converges uniformly to \( f \).

7. Let \( f \) be a continuous function on \((0, 1)\) taking values in \([0, 1]\) such that \( f(x) \leq x(1-x) \forall x \in (0, 1) \).

Then \( f \) is uniformly continuous on \((0, 1)\).

8. Let \( g \) be defined by \( g(x) = |x|^3 \exp\{-|x|\} \quad x \in \mathbb{R} \).

Then \( g \) is a continuously differentiable function on \( \mathbb{R} \).

9. Suppose \( \lambda^2 \) is an eigenvalue of \( A^2 \). Then \( \lambda \) is an eigenvalue of \( A \).
Part B
Answer any four questions. Each question carries 10 marks. State precisely any theorem that you use.

1. Let \( x, y \in \mathbb{R}^n \) be such that
\[
\|x + ty\| \geq \|x\|, \quad \forall t \in \mathbb{R}.
\]
Show that \( x \cdot y = 0 \).

2. Let \( p(x) \) be a \( n^{th} \) degree polynomial such that the equation \( p(x) = 0 \) admits \( n \) distinct real roots \( c_1, c_2, \ldots, c_n \). Suppose that \( p(x) \neq 0 \) for \( -1 < x < 1 \). Show that \( |c_j| \leq |p(0)| \) for \( j = 1, 2, \ldots, n \).

3. Let \( f(x) = |x| \exp \{-x\} \) for \( -1 \leq x \leq 1 \). Find \( u, v \in [-1, 1] \) such that
\[
f(u) \leq f(x) \leq f(v), \quad \forall x \in [-1, 1].
\]

4. Let \( A \) be a \( n \times n \) matrix and \( y \) be a \( n \times 1 \) matrix (vector) such that the equation
\[
Ax = y
\]
for a \( n \times 1 \) matrix (vector) \( y \) admits no solution. Show that the rank of \( A \) is strictly less than \( n \).

5. Let \( A = (a_{ij}) \) be a \( 100 \times 100 \) matrix defined by
\[
a_{ij} = i^2 + j^2.
\]
Find the rank of \( A \).

6. For \( n \geq 1 \) let
\[
a_n = \frac{(\log n)^4}{n^2}.
\]
Show that the series
\[
\sum_{n=1}^{\infty} a_n
\]
converges.
SOLUTIONS: PART A

Solutions, in brief, are provided. However, other correct solutions are also admitted.

Q1: False.
Take all $a_n = 1$. The power series converges for $|t| < 1$, so $R > 0$. But $a_n \not\to 0$.

Q2: True.
Take the vector $v$ with +1 at $j$-th coordinate; −1 at $k$-th coordinate and other coordinates zero. Since $A$ is positive definite, $\sum a_{mn}v_mv_n \geq 0$. Note $v_m v_n$ equals +1 if $m = n = j$ or $m = n = k$; equals −1 if $m = j, n = k$ or $m = k, n = j$. It is zero for all other values of $m$ and $n$. Since the matrix is symmetric $a_{jk} = a_{kj}$. We get $a_{jj} + a_{kk} - 2a_{jk} \geq 0$.

Q3: False.
First statement is true. If $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$, then
$q(x) = c_1 (x - a) + c_2 (x^2 - a^2) + \cdots + c_k (x^k - a^k)$
where each term has factor $(x - a)$. So $q$ has factor $(x - a)$.

Second statement false: Take $p(x) = (x - a)^2 + 5$, then $p(a) = 5 > 0$ but $q(x) = (x - a)^2$.

Q4: False.
True for $f$, false for $g$. For $x > 0$, $f(x) = x \sin x$. Since both $x$ and $\sin x$ are differentiable, so is $f$. Similarly it is differentiable for $x < 0$. When $x = 0$, $\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} \sin h$. First factor is ±1 and second factor converges to zero as $h \to 0$. So $f$ is differentiable at $x = 0$ and the derivative is zero.

Using same argument as above, $g$ is differentiable at all points $x \neq 0$. At $x = 0$, $\frac{g(0+h) - g(0)}{h} = \frac{|h|}{h} \cos h$ which equals $\cos h$ for $h > 0$ and equals $-\cos h$ for $h < 0$. So converges to +1 as $h > 0, h \to 0$; converges to −1 as $h < 0, h \to 0$. Thus the function is not differentiable at $x = 0$.

Q5: False.
Take $n = 2$, $A$ the $2 \times 2$ diagonal matrix with diagonal entries one. $B$ the $2 \times 2$ matrix with diagonal entries 2 and off diagonal entries 1. Both are positive definite, $a_{ij} < b_{ij}$. $C$ is the matrix with all entries −1. Since the first diagonal entry is negative it can not be positive definite.
Q6: False.

Take \( f_n(x) = nx \) for \( 0 \leq x \leq 1/n \); \( f_n(x) = n(\frac{2}{n} - x) \) for \( 1/n \leq x \leq 2/n \) and \( f_n(x) \) equals zero for \( x \geq 2/n \). All the functions are zero when \( x = 0 \). If \( x > 0 \) then \( x > 2/n \) for all large \( n \) so that \( f_n(x) = 0 \) for all large \( n \). Thus the functions converge to the identically zero function which is continuous. However the \( n \)-th function takes the value one at \( x = 1/n \), so the convergence is NOT uniform.

Q7: True.

Since \( 0 \leq f(x) \leq x(1 - x) \) we conclude that \( 0 \leq f(x) \leq x \) and so limit of \( f \) as \( x \to 0 \) exists and equals zero. Similarly \( 0 \leq f(x) \leq (1 - x) \) so that limit of \( f \) as \( x \to 1 \) exists and equals zero. \( f \) is given to be continuous on \((0, 1)\) and has limits zero as \( x \) approaches zero or one. Thus \( f \) can be extended to a continuous function on the closed bounded interval \([0, 1]\), so \( f \) is uniformly continuous.

Q8: True.

When \( x > 0 \), \( g(x) = x^3e^{-x} \), product of two differentiable functions; \( g'(x) = 3x^2e^{-x} - x^3e^{-x} \). Note that it is continuous on \((0, \infty)\) and converges to zero as \( x > 0, x \to 0 \). Similarly \( g(x) = -x^3e^x \) is differentiable for \( x < 0 \) and \( g'(x) = -3x^2e^x - x^3e^x \). Note that it is continuous on \((-\infty, 0)\) and converges to zero as \( x < 0, x \to 0 \).

At \( x = 0 \),
\[
\frac{f(0 + h) - f(0)}{h} = \frac{|h|^3e^{-|h|}}{h} = h|e^{-|h|} = 0
\]
as \( h \to 0 \). So the function is differentiable at \( x = 0 \) and the derivative is zero. Thus \( g'(0) = 0 \). Since we already saw that \( g'(x) \) converges to zero as \( x \neq 0, x \to 0 \); we conclude that \( g \) is continuously differentiable.

Q9: False.

Let \( A \) be the \( 2 \times 2 \) identity matrix (one on diagonals and zero off diagonals). Then \( 1 = (-1)^2 \) is an eigen value of \( A^2 = I \) but \((-1) \) is not an eigen value of \( A \).

SOLUTIONS PART B.

Q1. \( ||x + ty||^2 = \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2t\langle x, y \rangle + t^2\langle y, y \rangle \geq \langle x, x \rangle \). Thus \( t^2\langle y, y \rangle + 2t\langle x, y \rangle \geq 0 \) for all \( t \). Let \( \epsilon = ||\langle x, y \rangle||/(y, y) \). Now if \( \langle x, y \rangle > 0 \), then \( t = -\epsilon \) would yield a contradiction, while if \( \langle x, y \rangle < 0 \), then \( t = \epsilon \) would yield a contradiction.
Q2: If the leading coefficient of the polynomial is one, then the polynomial \( p \) must be 

\[
p(x) = (x - c_1)(x - c_2) \cdots (x - c_n)
\]

with \( |c_i| \geq 1 \) for each \( i \). So \( |p(0)| = |c_1c_2 \cdots c_n| \geq |c_j| \) for each \( j \). However this is not true if the leading coefficient is not one as the example 

\[
p(x) = (x - 2)/10
\]

shows.

(Candidates have been rewarded for any correct argument.)

Q3. For \( 0 \leq x \leq 1 \), 

\[
f(x) = xe^{-x}
\]

and its derivative 

\[
f'(x) = (1 - x)e^{-x} \geq 0,
\]

we see \( f \) is increasing, its maximum value is at \( x = 1 \) and minimum value is at zero. For \( -1 \leq x \leq 0 \), 

\[
f(x) = -xe^{-x},
\]

its derivative 

\[
f'(x) = (x - 1)e^{-x} \leq 0,
\]

so \( f \) is decreasing and its maximum value is at \( x = -1 \) and minimum value is at zero. Of these two maximum values 

\[
f(-1) = e > e^{-1} = f(1).
\]

The two minimum values are \( f(0) = 0 \). Thus 

\[
f(0) \leq f(x) \leq f(-1)
\]

for all \( x \) in this interval.

Q4: If rank of \( A \) is \( n \), then it is non-singular, invertible and so \( Ax = y \) has solution 

\[
x = a^{-1}y.
\]

Or, if the rank is \( n \), then the columns are linearly independent, so the \( n \) columns form a basis of \( R^n \) and so \( y \) is a linear combination of columns of \( A \) and so \( Ax = y \) has a solution.

Q5: Let \( B = ((b_{ij})) \) be defined by 

\[
b_{1j} = a_{1j} = 1 + j^2 \quad \text{and for } i \geq 2, \quad b_{ij} = a_{ij} - a_{i-1,j} = 2i - 1.
\]

Then it is clear that we can get \( B \) by suitable row operations on \( A \) (Subtract \( (i - 1)^{th} \) row from \( i^{th} \) row for \( i = n, n - 1, \ldots, 2 \)). Dividing \( i^{th} \) row by \( 2i - 1 \) for \( i \geq 2 \), and denoting the resulting matrix by \( C \), we get \( C = ((c_{ij})) \) with 

\[
c_{11} = 1 \quad \text{and } c_{ij} = 0 \quad \text{for } i \geq 2, \quad j \geq 2
\]

and clearly \( \text{rank}(B) = \text{rank} \). Thus for the matrix \( C \), the rows from 2 to \( n \) are identical and row 1 is independent of row 2. Thus rank of \( C \) is 2 and so the rank of \( A \) is 2.

Q6: For any positive integer \( k \), we have 

\[
e^x \geq x^{(k+1)/(k+1)!} \quad \text{so that } x^k e^{-x} \leq (k+1)!/x \to 0
\]

as \( x \to \infty \). In particular, taking \( k = 4 \) and the sequence 

\[
x_n = \log n^2/2 \to \infty
\]

we see 

\[
(\log n)^4/\sqrt{n} \to 0.
\]

Hence this convergent sequence is bounded by a number \( c \). So for the given series of positive terms, 

\[
\sum \log n/n^2 \leq \sum c/n^{3/2}
\]

a convergent series. Hence the given series is convergent.