CHENNAI Mathematical Institute

MSc Applications of Mathematics

Entrance Examination, 2012

Part A

Answer any 6 questions. Each question carries 10 marks. For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that the radius of convergence R of the power series $p(t) = \sum_{m=0}^{\infty} a_n t^n$ satisfies $R > 0$. The a_n converges to 0.

2. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix and suppose that A is positive definite. Then

$$a_{jk} \leq \tfrac{1}{2}(a_{jj} + a_{kk}).$$

3. Let $p(x)$ be a polynomial and $a > 0$ be such that $p(a) > 0$. Let $q(x) = p(x) - p(a)$. Then $(x - a)$ is a factor of $q(x)$ but $(x - a)^2$ is not a factor of $q(x)$.

4. Let $f(x) = |x| \sin(x)$ and $g(x) = |x| \cos(x)$ for $-2\pi \leq x \leq 2\pi$. Then f and g are differentiable on $[-2\pi, 2\pi]$.

5. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be $n \times n$ matrices that are positive definite such that

$$a_{ij} < b_{ij} \;\; \forall i, j.$$

   Let $\mathbf{C} = (c_{ij})$ be defined by $c_{ij} = a_{ij} - b_{ij}$. Then $\mathbf{C}$ is also positive definite.

6. Let f_n be a sequence of continuous functions on $[0, 1]$ converging to f point wise, where f is a continuous function. Then f_n converges uniformly to f.

7. Let f be a continuous function on $(0, 1)$ taking values in $[0, 1]$ such that

$$f(x) \leq x(1 - x) \quad \forall x \in (0, 1).$$

   Then f is uniformly continuous on $(0, 1)$.

8. Let g be defined by

$$g(x) = |x|^3 \exp\{-|x|\} \;\; x \in \mathbb{R}.$$

   Then g is a continuously differentiable function on $\mathbb{R}$.

9. Suppose λ^2 is an eigenvalue of $\mathbf{A}^2$. Then λ is an eigenvalue of $\mathbf{A}$.

Part B

Answer any four questions. Each question carries 10 marks.State precisely any theorem that you use.

1. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be such that
$$\|\mathbf{x} + t\mathbf{y}\| \geq \|x\|, \quad \forall t \in \mathbb{R}.$$
Show that $\mathbf{x} \cdot \mathbf{y} = 0$.

2. Let $p(x)$ be a n^{th} degree polynomial such that the equation $p(x) = 0$ admits n distinct real roots $c_1, c_2, \ldots c_n$. Suppose that $p(x) \neq 0$ for $-1 < x < 1$. Show that $|c_j| \leq |p(0)|$ for $j = 1, 2, \ldots n$.

3. Let $f(x) = |x| \exp\{-x\}$ for $-1 \leq x \leq 1$. Find $u, v \in [-1, 1]$ such that
$$f(u) \leq f(x) \leq f(v), \quad \forall x \in [-1, 1].$$

4. Let $\mathbf{A}$ be a $n \times n$ matrix and $\mathbf{y}$ be a $n \times 1$ matrix (vector) such that the equation
$$\mathbf{A}\mathbf{x} = \mathbf{y}$$
for a $n \times 1$ matrix (vector) $\mathbf{y}$ admits no solution. Show that the rank of $\mathbf{A}$ is strictly less than n.

5. Let $\mathbf{A} = (a_{ij})$ be a 100×100 matrix defined by
$$a_{ij} = i^2 + j^2.$$
Find the rank of $\mathbf{A}$.

6. For $n \geq 1$ let
$$a_n = \frac{(\log n)^4}{n^2}.$$
Show that the series
$$\sum_{n=1}^{\infty} a_n$$
converges.

SOLUTIONS: PART A

Solutions, in brief, are provided. However, other correct solutions are also admitted.

Q1: False.

Take all $a_n = 1$. The power series converges for $|t| < 1$, so $R > 0$. But $a_n \not\to 0$.

Q2: True.

Take the vector v with $+1$ at j-th coordinate; -1 at k-th coordinate and other coordinates zero. Since A is positive definite $\sum a_{mn} v_m v_n \geq 0$. Note $v_m v_n$ equals $+1$ if $m = n = j$ or $m = n = k$; equals -1 if $m = j, n = k$ or $m = k, n = j$. It is zero for all other values of m and n. Since the matrix is symmetric $a_{jk} = a_{kj}$. We get $a_{jj} + a_{kk} - 2a_{jk} \geq 0$.

Q3: False.

First statement is true. If $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$, then
$q(x) = c_1(x-a) + c_2(x^2 - a^2) + \cdots + c_k(x^k - a^k)$
where each term has factor $(x-a)$. So q has factor $(x-a)$.

Second statement false: Take $p(x) = (x-a)^2 + 5$, then $p(a) = 5 > 0$ but $q(x) = (x-a)^2$.

Q4: False.

True for f, false for g. For $x > 0$, $f(x) = x \sin x$. Since both x and $\sin x$ are differentiable, so is f. Similarly it is differentiable for $x < 0$. When $x = 0$, $\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} \sin h$. First factor is ± 1 and second factor converges to zero as $h \to 0$. So f is differentiable at $x = 0$ and the derivative is zero.

Using same argument as above, g is differentiable at all points $x \neq 0$. At $x = 0$, $\frac{g(0+h)-g(0)}{h} = \frac{|h|}{h} \cos h$ which equals $\cos h$ for $h > 0$ and equals $-\cos h$ for $h < 0$. So converges to $+1$ as $h > 0, h \to 0$; converges to -1 as $h < 0, h \to 0$. Thus the function is not differentiable at $x = 0$.

Q5: False.

Take $n = 2$, A the 2×2 diagonal matrix with diagonal entries one. B the 2×2 matrix with diagonal entries 2 and off diagonal entries 1. Both are positive definite, $a_{ij} < b_{ij}$. C is the matrix with all entries -1. Since the first diagonal entry is negative it can not be positive definite.

Q6: False.

Take $f_n(x) = nx$ for $0 \le x \le 1/n$; $f_n(x) = n(\frac{2}{n} - x)$ for $1/n \le x \le 2/n$ and $f_n(x)$ equals zero for $x \ge 2/n$. All the functions are zero when $x = 0$. If $x > 0$ then $x > 2/n$ for all large n so that $f_n(x) = 0$ for all large n. Thus the functions converge to the identically zero function which is continuous. However the n-th function takes the value one at $x = 1/n$, so the convergence is NOT uniform.

Q7: True.

Since $0 \le f(x) \le x(1-x)$ we conclude that $0 \le f(x) \le x$ and so limit of f as $x \to 0$ exists and equals zero. Similarly $0 \le f(x) \le (1-x)$ so that limit of f as $x \to 1$ exists and equals zero. f is given to be continuous on $(0,1)$ and has limits zero as x approaches zero or one. Thus f can be extended to a continuous function on the closed bounded interval $[0,1]$, so f is uniformly continuous.

Q8: True.

When $x > 0$, $g(x) = x^3 e^{-x}$, product of two differentiable functions; $g'(x) = 3x^2 e^{-x} - x^3.e^{-x}$. Note that it is continuous on $(0,\infty)$ and converges to zero as $x > 0, x \to 0$. Similarly $g(x) = -x^3 e^x$ is differentiable for $x < 0$ and $g'(x) = -3x^2 e^x - x^3 e^x$. Note that it is continuous on $(-\infty, 0)$ and converges to zero as $x < 0, x \to 0$.

At $x = 0$,

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|^3 e^{-|h|}}{h} = h|h|e^{-|h|} \to 0$$

as $h \to 0$. So the function is differentiable at $x = 0$ and the derivative is zero. Thus $g'(0) = 0$. Since we already saw that $g'(x)$ converges to zero as $x \ne 0, x \to 0$; we conclude that g is continuously differentiable.

Q9: False.

Let A be the 2×2 identity matrix (one on diagonals and zero off diagonals). Then $1 = (-1)^2$ is an eigen value of $A^2 = I$ but (-1) is not an eigen value of A.

SOLUTIONS PART B.

Q1. $||x + ty||^2 = \langle x + ty, x + ty\rangle = \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle \ge \langle x, x\rangle$. Thus $t^2\langle y, y\rangle + 2t\langle x, y\rangle \ge 0$ for all t. Let $\epsilon = |\langle x, y\rangle|/\langle y, y\rangle$. Now if $\langle x, y\rangle > 0$, then $t = -\epsilon$ would yield a contradiction, while if $\langle x, y\rangle < 0$, then $t = \epsilon$ would yield a contradiction.

Q2: If the leading coefficient of the polynomial is one, then the polynomial p must be $p(x) = (x - c_1)(x - c_2) \cdots (x - c_n)$ with $|c_i| \geq 1$ for each i. So $|p(0)| = |c_1 c_2 \cdots c_n| \geq |c_j|$ for each j. However this is not true if the leading coefficient is not one as the example $p(x) = (x - 2)/10$ shows.
(Candidates have been rewarded for any correct argument.)

Q3. For $0 \leq x \leq 1$, $f(x) = xe^{-x}$ and its derivative $f'(x) = (1 - x)e^{-x} \geq 0$, we see f is increasing, its maximum value is at $x = 1$ and minimum value is at zero. For $-1 \leq x \leq 0$, $f(x) = -xe^{-x}$, its derivative $f'(x) = (x - 1)e^{-x} \leq 0$, so f is decreasing and its maximum value is at $x = -1$ and minimum value is at zero. Of these two maximum values $f(-1) = e > e^{-1} = f(1)$. The two minimum values are $f(0) = 0$. Thus $f(0) \leq f(x) \leq f(-1)$ for all x in this interval.

Q4: If rank of A is n, then it is non-singular, invertible and so $Ax = y$ has solution $x = a^{-1}y$. Or, if the rank is n, then the columns are linearly independent, so the n columns form a basis of R^n and so y is a linear combination of columns of A and so $Ax = y$ has a solution.

Q5: Let $B = ((b_{ij}))$ be defined by $b_{1j} = a_{1j} = 1+j^2$ and for $i \geq 2$, $b_{ij} = a_{ij} - a_{i-1,j} = 2i-1$. Then it is clear that we can get B by suitable row operations on A (Subtract $(i-1)^t h$ row from i^{th} row for $i = n, n-1, \ldots, 2$). Dividing i^{th} row by $2i - 1$ for $i \geq 2$, and denoting the resulting matrix by C, we get $C = ((c_{ij}))$ with $c_{i1} = 1$ and $c_{ij} = 0$ for $i \geq 2, j \geq 2$ and clearly rank(B)=rank. Thus for the matrix C, the rows from 2 to n are identicle and row 1 is independent of row 2. Thus rank of C is 2 and so the rank of A is 2.

Q6: For any positive integer k, we have $e^x \geq x^{(k+1)}/(k+1)!$ so that $x^k e^{-x} \leq (k+1)!/x \to 0$ as $x \to \infty$. In particular, taking $k = 4$ and the sequence $x_n = \log n/2 \to \infty$, we see $(\log n)^4/\sqrt{n} \to 0$. Hence this convergent sequence is bounded by a number c. So for the given series of positive terms, $\sum \log n/n^2 \leq \sum c/n^{3/2}$ a convergent series. Hence the given series is convergent.