

**CHENNAI Mathematical Institute**  
**MSc Applications of Mathematics**

Notation:  $\mathbb{R}$  denotes the set of real numbers.

**Part A**

Answer all questions.

For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let  $p$  be a prime number such that  $p$  divides  $n^2$  for an integer  $n$ . Then  $p^2$  also divides  $n^2$ .
2. Let  $p(x)$  be a polynomial of odd-degree with real coefficients. Then the equation  $p(x) = 0$  admits at least one real root.
3. For positive real numbers  $a, b$  and positive integers  $n$ ,

$$a^{2n} + b^{2n} \leq 2a^n b^n.$$

4. Let  $a_n = \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$ . Then  $\{a_n\}$  is a decreasing sequence.
5. For a positive integer  $n > 1$ ,

$$\sum_{k=1}^{n-1} \binom{n}{k} = 2(2^{n-1} - 1).$$

6. Let  $f(x) = |x|^3$  for  $x \in \mathbb{R}$ . Then  $f$  is differentiable at  $x = 0$ .
7. Let  $\{a_n\}, \{b_n\}$  be sequences of strictly positive real numbers such that  $\{a_n\}$  converges to zero and  $\{b_n\}$  is bounded. Then  $\{c_n\}$  also converges to zero where  $c_n = a_n b_n$ .
8. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded continuous function. Then  $f$  attains its supremum.
9. Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times n$ - matrices such that  $\mathbf{ABABA} = \mathbf{I}_n$  where  $\mathbf{I}_n$  denotes  $n \times n$  identity matrix. Then  $\mathbf{B}$  is invertible.
10. Let  $\mathbf{A}$  be real symmetric matrix such that  $\mathbf{A}^3 = \mathbf{A}$ . If  $\lambda$  is an eigenvalue of  $\mathbf{A}$  then either  $\lambda = 0$  or  $\lambda = 1$ .

**Part B**

Answer all questions.

1. Let  $f$  be a one-one differentiable function from  $\mathbb{R}$  onto  $\mathbb{R}$  such that  $f(0) < f(1)$ . Show that  $f$  is a strictly increasing function.
2. Let  $p(x) = ax^2 + 2bx + c$  be a quadratic form, where  $a, b, c$  are real numbers. Find supremum of  $p(x)$  over  $x \in \mathbb{R}$ .
3. Let  $\{c_1, c_2, \dots, c_n\}$  and  $\{d_1, d_2, \dots, d_n\}$  be non-zero real numbers and let  $a_{ij} = c_i d_j$ . Find the rank of the matrix  $A = (a_{ij})$ .
4. Let  $A$  be a  $n \times n$  real symmetric positive definite matrix. Show that the eigenvalues of  $A$  are strictly positive.
5. Show that every symmetric  $n \times n$  matrix  $A$  with real entries can be written as  $A = B - C$  where  $B, C$  are positive definite symmetric matrices.
6. Let  $f$  be a function on  $\mathbb{R}$  such that  $f'(x) = f(x)$  for all  $x \in \mathbb{R}$  where  $f'$  denotes the derivative of  $f$  and  $f(0) = 1$ . Show that  $f(x) = \exp(x)$  for all  $x$ .
7. Let  $f(x) = x^2$  for  $x \geq 0$  and  $f(x) = -x^2$  for  $x < 0$ . Is  $f$  differentiable at  $x = 0$ ? Give reasons for your answer.
8. 10 balls numbered 1 to 10 are to be distributed in 10 boxes. Find number of ways of distributing the balls so that exactly one box remains empty.
9. Let  $\mathbf{A}$  be a  $m \times n$  matrix such that for every  $m \times 1$  vector  $\mathbf{y}$  the equation  $\mathbf{Ax} = \mathbf{y}$  has a solution. Show that  $n \geq m$ .
10. Give an example of a continuous function from the set of real numbers to itself that is not differentiable at 0 and 1; and is differentiable at all other points.