NCM IST, Mathematics for Computer Science
Systems of linear equations

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1. A matrix A is said to be totally unimodular if every square submatrix of A has determinant 0 or ± 1. Let A be a totally unimodular matrix. Suppose we construct a new matrix $\tilde{A}$ from A by appending a unit vector e_i as a last new column. Show that $\tilde{A}$ is also totally unimodular.

2. Consider a linear program with n non negative variables and m inequalities

$$\begin{array}{ll} max & c^t x \\ \text{s.t} & Ax \leq b \\ & \;\; x \;\; \geq 0 \end{array}$$

   If A is totally unimodular, the entries of b are integral and if the linear program has an optimal solution, show that it also has an integral optimal solution.

3. Look at the LP formulation of the maximum cardinality matching on bipartite graphs Show that there is an integral optimal and thereby construct a matching. Write down the dual and argue that it has an integral optimal. What does that correspond to?

4. We are given a collection of points in the plane (x_i, y_i), $i = 1, \dots, n$. We wish to find a line $y = ax + b$ so that the sum of the squares of the distances to the line is as small as possible. Set up the optimization problem. Set it up in the case when you minimize the sum of the modulus of the distances to the line instead of the square.

5. **Farkas's lemma** Let $A \in \mathbb{R}^{mxn}$, Then exactly one of the following statements is true:

    1. There exist a $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$
    2. There exist a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$, and $b^T y < 0$

   Assume statement one is false. Show that statement 2 is true. In the class we have seen $A^T y \geq 0$. You need to show that $b^T y < 0$.