Reconciling Weighted MSO and Probabilistic CTL

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Invited talk at DLT'09

Motivations

Analysis of quantitative systems

- Probabilistic Systems
- Minimization of costs
- Maximization of rewards
- Computation of reliability
- Optimization of energy consumption
- . . .

Models (no time)

- Probabilistic automata (generative, reactive)
- Transition systems with costs or rewards
- . . .

All are special cases of Weighted Automata.

Motivations

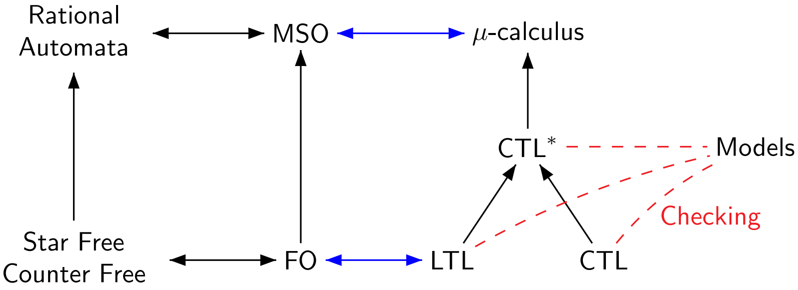
Specification

- PCTL: Probabilistic CTL — Hansson & Jonsson, '94
- PCTL*: Probabilistic CTL* — de Alfaro, '98
- CTL$: Valued CTL — Buchholz & Kemper, '03, '09
- wMSO: Weighted MSO — Droste & Gastin, '05, '07, '09

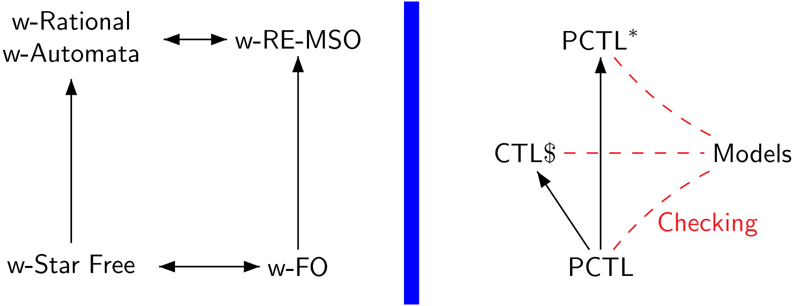
Natural Problems

- Satisfiability
- Model Checking
- Expressivity

Qualitative (Boolean) Picture



Quantitative Picture



Our aim is to compare and unify these logics

Plan

1. Weighted Automata

Weighted MSO Logic

Weighted CTL^* **and** PCTL^*

Weighted CTL^* **versus weighted** MSO

Conclusion and Open problems

Semirings

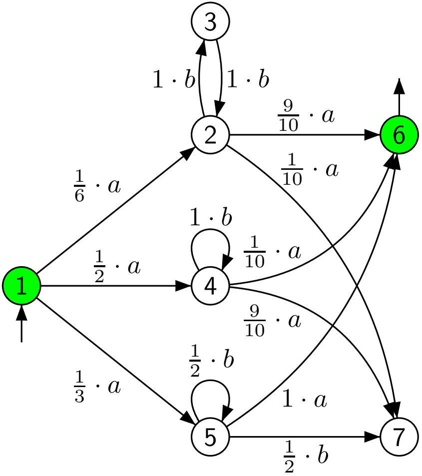
Definition: Semiring

- $\mathbb{K} = (K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$
- $(K, \oplus, \mathbf{0})$ is a commutative monoid,
- $(K, \otimes, \mathbf{1})$ is a monoid,
- multiplication distributes over addition, and $\mathbf{0}$ is absorbant.

Examples:

- Boolean: $\mathbb{B} = (\{\mathbf{0}, \mathbf{1}\}, \vee, \wedge, \mathbf{0}, \mathbf{1})$
- Natural: $(\mathbb{N}, +, \cdot, 0, 1)$
- Tropical: $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$
- Probabilistic: $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$
- Reliability: $([0, 1], \max, \cdot, 0, 1)$

Weighted Automata by Examples



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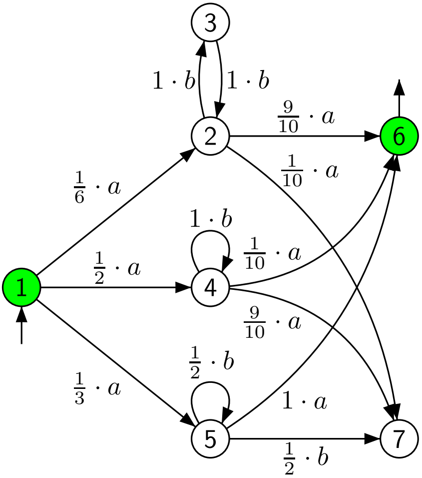

Several paths for $v = ab^n a$:

$\pi_1 = 1 \xrightarrow{a} 4 \xrightarrow{b} 4 \cdots 4 \xrightarrow{b} 4 \xrightarrow{a} 6$
$\text{weight}(\pi_1) = \frac{1}{2} \cdot 1^n \cdot \frac{1}{10} = \frac{1}{20}$

$\pi_2 = 1 \xrightarrow{a} 5 \xrightarrow{b} 5 \cdots 5 \xrightarrow{b} 5 \xrightarrow{a} 6$
$\text{weight}(\pi_2) = \frac{1}{3} \cdot (\frac{1}{2})^n \cdot 1 = \frac{1}{3 \cdot 2^n}$

If n is even:
$\pi_3 = 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \cdots 2 \xrightarrow{a} 6$
$\text{weight}(\pi_3) = \frac{1}{6} \cdot 1^n \cdot \frac{9}{10} = \frac{3}{20}$

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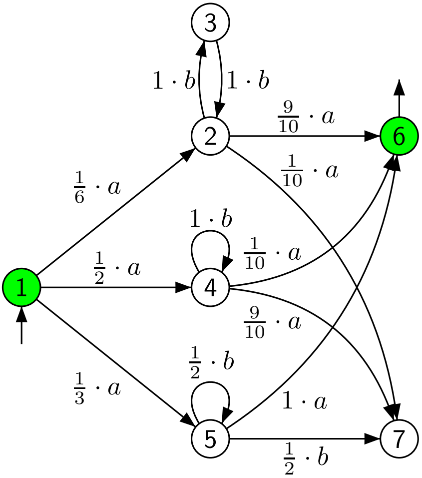

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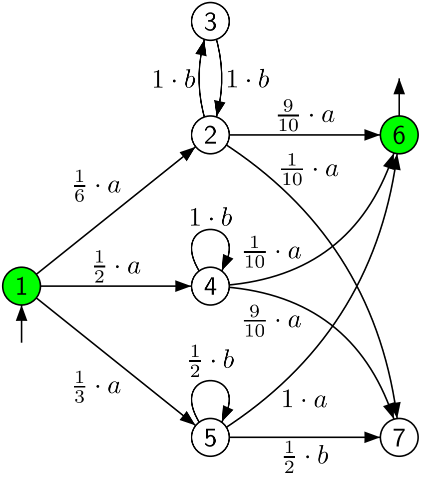
Probabilistic: $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

$$[\![\mathcal{A}]\!](v) = \begin{cases} \frac{1}{20} + \frac{1}{3 \cdot 2^n} & \text{if } n \text{ is odd} \\ \frac{1}{5} + \frac{1}{3 \cdot 2^n} & \text{if } n \text{ is even} \end{cases}$$

Weighted Automata by Examples



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Reliability: $([0,1], \max, \cdot, 0, 1)$

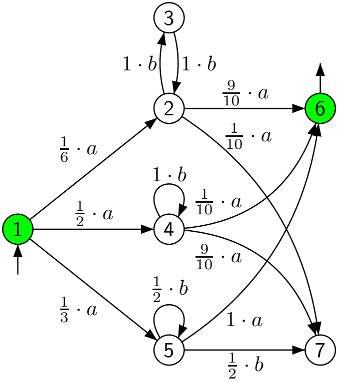
$$[\![\mathcal{A}]\!](v) = \begin{cases} \max(\frac{1}{20}, \frac{1}{3 \cdot 2^n}) & \text{if } n \text{ is odd} \\ \max(\frac{3}{20}, \frac{1}{3 \cdot 2^n}) & \text{if } n \text{ is even} \end{cases}$$

Reactive Probabilistic Finite Automata

Definition: rPFA on $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

A *reactive probabilistic finite automaton (*rPFA*)* is a weighted automaton $\mathcal{A} = (Q, q_0, \mu, F)$ over $\mathbb{P}\text{rob}$ such that, for all $q \in Q$ and $a \in \Sigma$,

$$\sum_{q' \in Q} \mu(q, a, q') \in \{0, 1\}$$

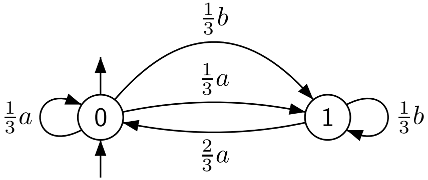


Generative Probabilistic Finite Automata

Definition: $\textsc{gPFA}$ on $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

A *generative probabilistic finite automaton ($\textsc{gPFA}$)* is a weighted automaton $\mathcal{A} = (Q, q_0, \mu, F)$ over $\mathbb{P}$rob such that, for all $q \in Q$,

$$\sum_{(a,q') \in \Sigma \times Q} \mu(q, a, q') \in \{0, 1\}$$



Plan

Weighted MSO

Short history

Introduced by Droste & Gastin (ICALP'05)
Aim: Logical characterization of weighted automata.
Generalization of Elgot's and Büchi's theorems.

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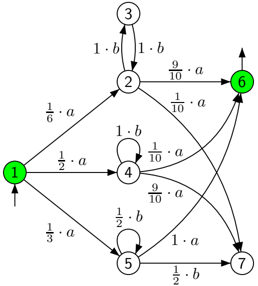
Extended to

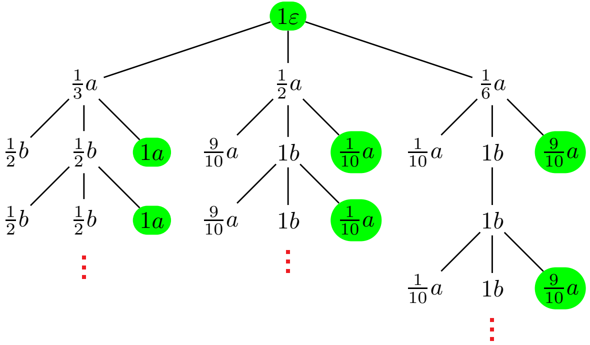
- Trees — Droste & Vogler
- Infinite words — Droste & Kuske, Droste & Rahonis
- Pictures — Fischtner
- Traces — Meinecke
- Distributed systems — Bollig & Meinecke
- ...

No link with quantitative temporal logics such as PCTL or CTL\$.

Weighted Trees

Semantics of weighted MSO is on weighted trees
which are unfoldings of weighted automata





Definition: Weighted Trees: $Trees(D, \mathbb{K}, \Sigma)$

$$\begin{array}{rccc} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \longrightarrow & (\kappa_t(u), \ell_t(u)) \end{array}$$

Extended Weighted MSO

Definition: Syntax of wMSO$(\mathbb{K}, \Sigma, \mathcal{C})$

$$\varphi ::= k \mid \kappa(x) \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)})$$
$$\mid P_a(x) \mid x \leq y \mid x \in X \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi$$

where $k \in K$, $a \in \Sigma$, x, y are first-order variables, X is a set variable and $\bowtie \in \mathcal{C}$.

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- $\mathcal{C}$ is a vocabulary of symbols $\bowtie \in \mathcal{C}$ with $\mathrm{arity}(\bowtie) \in \mathbb{N}$.
  - $\mathcal{C} = \{\vee, \wedge, \neg\}$
  - $\mathcal{C} = \{\wedge, \neg, \prec\}$

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  - $\mathcal{C} = \{\vee, \wedge, \neg\}$
  - $\mathcal{C} = \{\wedge, \neg, \prec\}$
- Each symbol $\bowtie \in \mathcal{C}$ is given a semantics $[\![\bowtie]\!] : K^{\mathrm{arity}(\bowtie)} \rightharpoonup K$.
  - $[\![\vee]\!] = \oplus$
  - $[\![\wedge]\!] = \otimes$
  - $[\![\neg]\!](k) = \begin{cases} \mathbf{1} & \text{if } k = \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}$
  - Probabilistic: $[\![\neg]\!](k) = 1 - k$ or $[\![\neg]\!](k) = \max(0, 1 - k)$
  - Ordered semiring: $[\![\prec]\!] : K^2 \to \{\mathbf{0}, \mathbf{1}\}$

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Definition: Semantics: $[\![\varphi]\!]_{\mathcal{V}} : \mathit{Trees}(D, \mathbb{K}, \Sigma_{\mathcal{V}}) \rightharpoonup K$

Let $\mathcal{V}$ be a finite set of first-order and second-order variables with $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$.

Let $t : \begin{array}{ccc} D^* & \rightharpoonup & K \times \Sigma \\ u & \longrightarrow & (\kappa_t(u), \ell_t(u)) \end{array}$ be a weighted tree and σ a $(\mathcal{V}, t)$-assignment.

$$[\![P_a(x)]\!]_{\mathcal{V}}(t, \sigma) = \begin{cases} \mathbf{1} & \text{if } \ell_t(\sigma(x)) = a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$[\![x \leq y]\!]_{\mathcal{V}}(t, \sigma) = \begin{cases} \mathbf{1} & \text{if } \sigma(x) \leq \sigma(y) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$\leq$ is the prefix ordering on $\mathrm{dom}(t)$

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$$[\![k]\!]_{\mathcal{V}}(t, \sigma) = k$$

$$[\![\kappa(x)]\!]_{\mathcal{V}}(t, \sigma) = \kappa_t(\sigma(x))$$

$$[\![\bowtie(\varphi_1, \ldots, \varphi_r)]\!]_{\mathcal{V}}(t, \sigma) = [\![\bowtie]\!]([\![\varphi_1]\!]_{\mathcal{V}}(t, \sigma), \ldots, [\![\varphi_r]\!]_{\mathcal{V}}(t, \sigma)) \qquad \text{if } \mathrm{arity}(\bowtie) = r$$

Recall that $[\![\vee]\!] = \oplus$ and $[\![\wedge]\!] = \otimes$

Extended Weighted MSO

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First Example

Example:

Let $\varphi_1 = \exists x.(P_b(x) \wedge (\kappa(x) > 0))$.

$$[\![\varphi_1]\!](t) = \bigoplus_{u \in \mathrm{dom}(t)} (\ell_t(u) = b) \otimes (\kappa_t(u) > 0)$$

is the number of nodes labeled b and having a positive weight.

Examples and Macros

Definition: Useful macro

$$\varphi_1 \xrightarrow{+} \varphi_2 \stackrel{\text{def}}{=} \neg\varphi_1 \vee (\varphi_1 \wedge \varphi_2)$$

If φ_1 is boolean (i.e., if $[\![\varphi_1]\!]$ takes values in $\{0, 1\}$), we have

$$[\![\varphi_1 \xrightarrow{+} \varphi_2]\!]_{\mathcal{V}}(t,\sigma) = \begin{cases} [\![\varphi_2]\!]_{\mathcal{V}}(t,\sigma) & \text{if } [\![\varphi_1]\!]_{\mathcal{V}}(t,\sigma) = \mathbf{1} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

If φ_1, φ_2 are boolean, then $\varphi_1 \xrightarrow{+} \varphi_2$ is the usual boolean implication.

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If φ_1, φ_2 are boolean, then $\varphi_1 \xrightarrow{+} \varphi_2$ is the usual boolean implication.

Example:

Let $\varphi_2 = \forall x.((P_a(x) \wedge (\kappa(x) > 0)) \xrightarrow{+} \kappa(x))$.

$$[\![\varphi_2]\!](t) = \bigotimes_{u \in \text{dom}(t)} ((P_a(u) \wedge (\kappa_t(u) > 0)) \xrightarrow{+} \kappa_t(u))$$

multiplies the positive values of a-labeled nodes.

Examples and Macros

Definition: Macros for Boolean formulas

$$\varphi_1 \underline{\vee} \varphi_2 \stackrel{\text{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$$

$$\underline{\exists} x.\varphi \stackrel{\text{def}}{=} \neg\forall x.\neg\varphi$$

$$\underline{\exists} X.\varphi \stackrel{\text{def}}{=} \neg\forall X.\neg\varphi$$

Hence, we can easily define boolean formulas for all MSO properties.

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Hence, we can easily define boolean formulas for all MSO properties.

Example:

- Let $\text{path}(x, X)$ be a boolean formula stating that X is a maximal path starting from node x,
- The following boolean formula checks if X satisfies a SU b, $\psi(x, X) = \underline{\exists} z.(z \in X \wedge x < z \wedge P_b(z) \wedge \forall y.(x < y < z \xrightarrow{+} P_a(y)))$
- The quantitative formula $\xi(x, X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$ computes the weight of path X, i.e., the product of weights of nodes in $X \setminus \{x\}$.

Then, we compute the sum of weights of paths from x satisfying a SU b with

$$\exists X.(\text{path}(x, X) \wedge \psi(x, X) \wedge \xi(x, X))$$

Original Weighted MSO

Definition: Original Weighted MSO — Droste & Gastin

- $\mathcal{C} = \{\vee, \wedge\}$
- negations over atomic formulas only
- models are unweighted finite words
- $\kappa(x)$ is not allowed

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Theorem: Droste & Gastin

- From any w-Aut $\mathcal{A}$ we can construct a formula φ in sREMSO s.t. $[\![\varphi]\!] = [\![\mathcal{A}]\!]$,
- From any formula φ in sREMSO we can construct a w-Aut $\mathcal{A}$ s.t. $[\![\varphi]\!] = [\![\mathcal{A}]\!]$.

sREMSO is a syntactic restriction of the existential fragment.

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sREMSO is a syntactic restriction of the existential fragment.

Definition: Satisfiability (for good semirings)

A formula φ is satisfiable if $[\![\varphi]\!](w) \neq \mathbf{0}$ for some word w.

Corollary: Satisfiability

The satisfiability problem is decidable for sREMSO.

Extended Weighted MSO

Proposition: Satisfiability

The satisfiability problem for wMSO($\mathbb{P}$rob$, \Sigma, \{\vee, \wedge, \neg, <\}$) is undecidable.

Proof:

Let $\mathcal{A} = (Q, q_0, \mu, F)$ be a reactive probabilistic finite automaton over Σ.

By [DG], $\exists \varphi \in \mathrm{sREMSO}(\mathbb{P}\mathrm{rob}, \Sigma, \{\vee, \wedge, \neg\})$ such that $[\![\varphi]\!](w) = [\![\mathcal{A}]\!](w)$ for all unweighted words $w \in \Sigma^*$.

Since φ does not use $\kappa(x)$, considering weighted or unweighted words or trees does not make any difference.

Now, for $p \in [0, 1]$ and $w \in \Sigma^*$ we have $[\![p < \varphi]\!](w) \neq 0$ iff $[\![\mathcal{A}]\!](w) > p$.

Hence, $p < \varphi$ is satisfiable iff the automaton $\mathcal{A}$ with threshold p accepts a nonempty language. By , A. Paz (1971) this is undecidable.

Plan

Weighted Automata

Weighted MSO Logic

3. Weighted CTL* and PCTL*

Weighted CTL* versus weighted MSO

Conclusion and Open problems

Weighted CTL*

Definition: Syntax of wCTL*$(\mathbb{K}, \mathit{Prop}, \mathcal{C})$

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \ \mathsf{SU}\ \psi$

Quantitative state formulas: $\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\psi)$

where $p \in \mathit{Prop}$, $k \in K$, $\bowtie \in \mathcal{C}$.

Weighted CTL*

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Quantitative state formulas: $\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\psi)$

where $p \in \mathit{Prop}$, $k \in K$, $\bowtie \in \mathcal{C}$.

Definition: Semantics for Boolean path formulas

$$t: \begin{array}{ccc} D^* & \rightharpoonup & K \times \Sigma \\ u & \to & (\kappa_t(u), \ell_t(u)) \end{array}$$

weighted tree, w branch of t, u node on w.

$t, w, u \models \varphi$ if $[\![\varphi]\!](t, u) \neq \mathbf{0}$

$t, w, u \models \psi_1 \wedge \psi_2$ if $t, w, u \models \psi_1$ and $t, w, u \models \psi_2$

$t, w, u \models \neg\psi$ if $t, w, u \not\models \psi$

$t, w, u \models \psi_1 \text{ SU } \psi_2$ if $\exists u < v \leq w : (t, w, v \models \psi_2$ and $\forall u < v' < v : t, w, v' \models \psi_1)$

Weighted CTL*

Definition: Syntax of wCTL$^*(\mathbb{K}, Prop, \mathcal{C})$

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \,\mathsf{SU}\, \psi$

Quantitative state formulas: $\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\text{arity}(\bowtie)}) \mid \mu(\psi)$

where $p \in Prop$, $k \in K$, $\bowtie \in \mathcal{C}$.

Definition: Semantics for quantitative state formulas

$$\begin{array}{llll} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \to & (\kappa_t(u), \ell_t(u)) \end{array}$$

weighted tree, u node of t, $\Sigma = 2^{Prop}$.

$$[\![k]\!](t,u) = k$$

$$[\![\kappa]\!](t,u) = \kappa_t(u)$$

$$[\![p]\!](t,u) = \begin{cases} \mathbf{1} & \text{if } p \in \ell_t(u) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$[\![\bowtie(\varphi_1, \ldots, \varphi_r)]\!](t,u) = [\![\bowtie]\!]([\![\varphi_1]\!](t,u), \ldots, [\![\varphi_r]\!](t,u)) \qquad \text{if arity}(\bowtie) = r$$

Weighted CTL*

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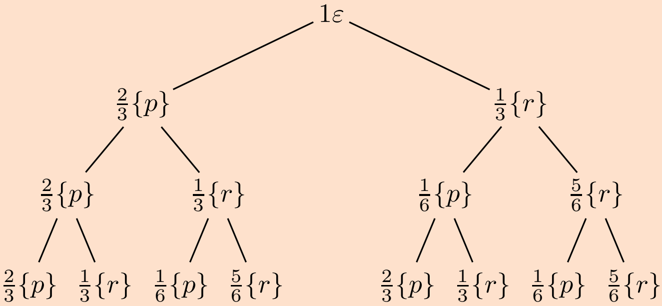
$$[\![\mu(\psi)]\!](t,u) = \bigoplus_{w \in \mathrm{Branches}(t) \mid t,w,u \models \psi} \; \bigotimes_{v \mid u < v \leq w} \kappa_t(v)$$

Example for $\mu(\psi)$ on a finite tree

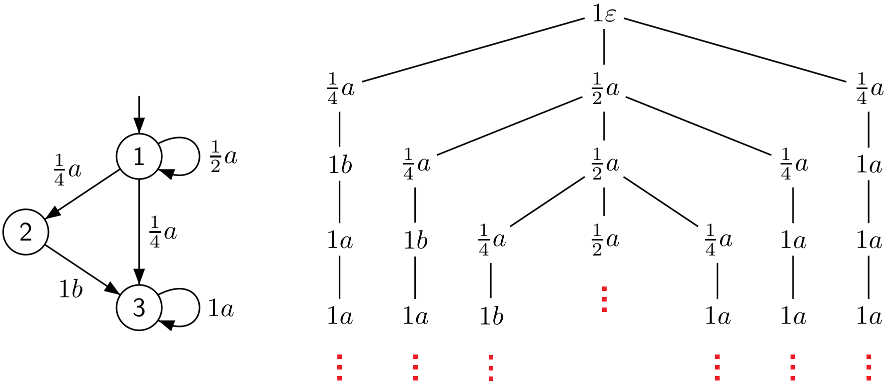
Example:

$$[\![\mu(\psi)]\!](t,u) = \bigoplus_{w \in \mathrm{Branches}(t) \,|\, t,w,u \models \psi} \; \bigotimes_{v \,|\, u < v \leq w} \kappa_t(v)$$



$$[\![\mu(p \,\mathsf{SU}\, r)]\!](t) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{1}{6}+\frac{5}{6}\right) + \frac{1}{3}\cdot(1) = \frac{19}{27}$$

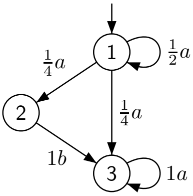
Unfoldings are infinite (regular) trees

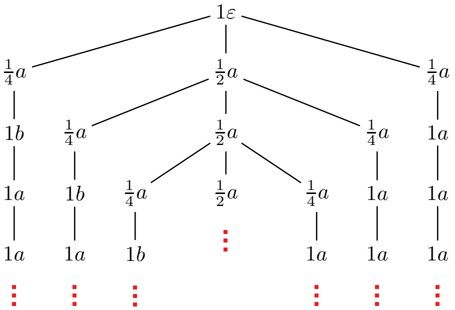


We need infinite sums and products

Example:

$$[\![\mu(\mathsf{F}\, b)]\!](t,\varepsilon) = \bigoplus_{w \text{ left branch}} \ \bigotimes_{v \mid \varepsilon < v \leq w} \kappa_t(v) = \sum_{n \geq 0} \frac{1}{2^n} \cdot \frac{1}{4} \cdot 1 = \frac{1}{2}$$



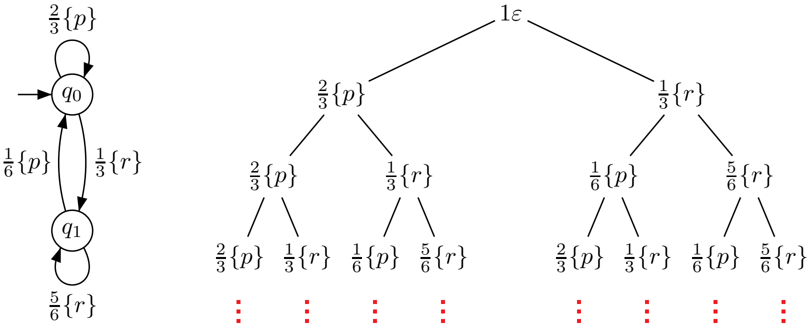


Infinite sums and products

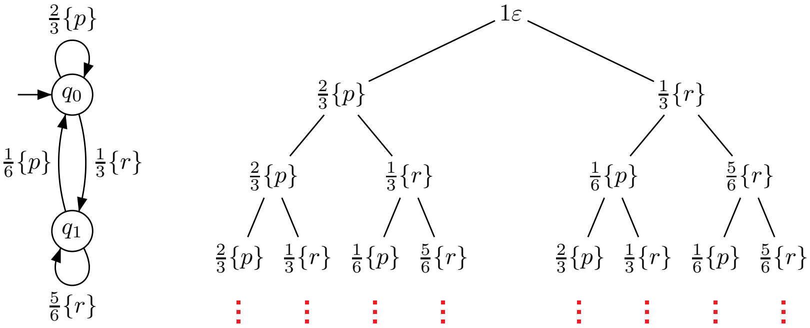
Some well-defined infinite sums or products

- $\bigoplus_{i \in I} k_i$ is well defined if $|\{i \in I \mid k_i \neq 0\}| < \infty$,
- $\bigotimes_{i \in I} k_i$ is well defined if $|\{i \in I \mid k_i \neq 1\}| < \infty$,
- $\bigotimes_{i \in I} k_i$ is well defined if $k_i = 0$ for some $i \in I$,
- $\sum_{i \geq 0} \frac{1}{2^i}$

Unfoldings of gPFA



Unfoldings of gPFA



Probability measure

- The weight of each branch is an infinite product which converges to 0.
- The sum of the weights of all branches starting from any node should be 1.
- To define $[\![\mu(\psi)]\!]$, we use the probability measure on the sequence space.
- We get $[\![\mu(p \,\mathsf{SU}\, r)]\!](t,\varepsilon) = \sum_{n\geq 0} \left(\frac{2}{3}\right)^n \cdot \frac{1}{3} = 1.$

PCTL* **is a boolean fragment of** wCTL*

Definition: Probabilistic computation tree logic PCTL* de Alfaro '98

The syntax of PCTL* is given by:

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \,\mathsf{SU}^{\leq n}\, \psi$

Boolean state formulas: $\varphi ::= 0 \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mu(\psi) \geq k \mid \mu(\psi) > k$

where $n \in \mathbb{N} \cup \{\infty\}$, $p \in Prop$, $k \in [0,1]$.

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Recall: Syntax of wCTL*$(\mathbb{P}\text{rob}, Prop, \{\neg, \wedge, \geq\})$

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi\ \mathsf{SU}\ \psi$

Quantitative state formulas: $\varphi ::= k \mid \kappa \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \geq \varphi \mid \mu(\psi)$

where $p \in Prop$, $k \in \mathbb{R}$.

Remark: PCTL* is a boolean fragment of wCTL*

State formulas are restricted:

- do not use κ,
- use $\geq$ and $\mu(\psi)$ only in comparisons of the form: $(\mu(\psi) \geq k)$ or $\neg(k \geq \mu(\psi))$

wCTL **is a fragment of** wCTL*

Definition: Syntax of wCTL($\mathbb{K}, \mathit{Prop}, \mathcal{C}$)

Only quantitative state formulas:

$$\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\varphi \ \mathsf{SU}^{\leq n}\ \varphi)$$

where $p \in \mathit{Prop}$, $k \in K$, $\bowtie \in \mathcal{C}$, $n \in \mathbb{N} \cup \{\infty\}$.

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Remark: wCTL is a fragment of wCTL$^*(\mathbb{K}, Prop, \mathcal{C})$

Boolean path formulas are restricted to $\psi ::= \varphi \,\mathsf{SU}^{\leq n}\, \varphi$

PCTL **is a fragment of** wCTL

Definition: Probabilistic CTL — Hansson & Jonsson '94

Only Boolean state formulas:

$$\varphi ::= 0 \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mu(\varphi \,\mathsf{SU}^{\leq n}\, \varphi) \geq k \mid \mu(\varphi \,\mathsf{SU}^{\leq n}\, \varphi) > k$$

where $n \in \mathbb{N} \cup \{\infty\}$, $p \in \mathit{Prop}$, $k \in [0,1]$.

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Remark: PCTL is a fragment of wCTL($\mathbb{P}$rob, $Prop$, $\{\neg, \wedge, \geq\}$)

Plan

Weighted Automata

Weighted MSO Logic

Weighted CTL* **and** PCTL*

4. Weighted CTL* versus weighted MSO

Conclusion and Open problems

wCTL^* **is a fragment of** wMSO

Theorem:

wCTL^* is a fragment of wMSO for finite trees and arbitrary semirings.

Proof: Translation of boolean path formulas

$$\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \;\mathsf{SU}\; \psi$$

Implicitely, ψ has two free variables, the path (set of nodes) and the current node.

We build a boolean MSO formula $\underline{\psi}(x, X) \in \mathsf{bMSO}(\mathbb{K}, \Sigma, \mathcal{C})$.

$$\underline{\varphi}(x, X) = (\overline{\varphi}(x) \neq \mathbf{0})$$

$$\underline{\psi_1 \wedge \psi_2}(x, X) = \underline{\psi_1}(x, X) \wedge \underline{\psi_2}(x, X)$$

$$\underline{\neg\psi}(x, X) = \neg\underline{\psi}(x, X)$$

$$\underline{\psi_1 \;\mathsf{SU}\; \psi_2}(x, X) = \underline{\exists} z.(z \in X \wedge x < z \wedge \underline{\psi_2}(z, X) \wedge \forall y.((x < y < z) \xrightarrow{+} \underline{\psi_1}(y, X)))$$

We assume that the interpretation of X is indeed a path.

We use $\underline{\exists}$, $\underline{\vee}$ and $\xrightarrow{+}$ to get boolean formulas.

wCTL* **is a fragment of** wMSO

Proof: Translation of quantitative state formulas

$$\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\psi)$$

Here, φ only has an implicit free variable, the current node.

We build a weighted MSO formula $\overline{\varphi}(x) \in \mathsf{bMSO}(\mathbb{K}, \Sigma, \mathcal{C})$.

$$[\![\mu(\psi)]\!](t,u) = \bigoplus_{w \in \mathrm{Branches}(t) \mid t,w,u \models \psi} \; \bigotimes_{v \mid u < v \leq w} \kappa_t(v)$$

$$\overline{\mu(\psi)}(x) = \exists X.(\mathrm{path}(x,X) \wedge \underline{\psi}(x,X) \wedge \xi(x,X))$$

$$\mathrm{path}(x,X) = x \in X$$

$$\wedge \forall z.(z \in X \xrightarrow{+} (z = x \,\underline{\vee}\, \underline{\exists} y.(y \in X \wedge y \lessdot z)))$$

$$\wedge \neg \underline{\exists} y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z')$$

$$\wedge \forall y.(\ (y \in X \wedge \underline{\exists} z.(y < z)) \xrightarrow{+} \underline{\exists} z.(z \in X \wedge y < z)\ )$$

$$\xi(x,X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$$

wCTL **is a fragment of** wMSO **on gPFA**

Theorem:

wCTL is a fragment of wMSO on probabilistic systems (GPFA).

Unfoldings of probabilistic systems (GPFA) are infinite.

The translation of $\overline{\mu(\psi)}(x)$ given above does not work.

We need to be careful with the induced infinite sums and products.

wCTL **is a fragment of** wMSO **on gPFA**

Proof: Translation of $\mu(\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2)$

$$\overline{\mu(\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2)}(x) = \exists X.(\mathrm{path}^{\leq n}(x, X) \wedge \underline{\psi}(x, X) \wedge \xi(x, X))$$

$$\begin{aligned}\mathrm{path}^{\leq \infty}(x, X) = {} & x \in X \\ & \wedge \forall z.(z \in X \xrightarrow{+} (z = x \,\underline{\vee}\, \underline{\exists}\, y.(y \in X \wedge y \lessdot z))) \\ & \wedge \neg \,\underline{\exists}\, y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z')\end{aligned}$$

if $n \in \mathbb{N}$, $\quad \mathrm{path}^{\leq n}(x, X) = \mathrm{path}^{\leq \infty}(x, X) \wedge \neg \,\underline{\exists}\, x_0 \ldots \underline{\exists}\, x_n.$
$$(x_0 \in X \wedge \cdots \wedge x_n \in X \wedge x < x_0 < x_1 < \cdots < x_n)$$

$$\psi = (\varphi_1 \wedge \neg\varphi_2) \,\mathsf{SU}\, (\varphi_2 \wedge \neg(\mathbf{0} \,\mathsf{SU}\, \mathbf{1}))$$

$$\xi(x, X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$$

$\mathrm{path}^{\leq n}(x, X) \wedge \underline{\psi}(x, X)$ is a boolean formula which holds if and only if X is a minimal path satisfying $\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2$.

wCTL **is a fragment of** wMSO **on gPFA**

Proof: Translation of $\mu(\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2)$

$$\overline{\mu(\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2)}(x) = \exists X.(\mathrm{path}^{\leq n}(x,X) \wedge \underline{\psi}(x,X) \wedge \xi(x,X))$$

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$$\text{if } n \in \mathbb{N}, \quad \mathrm{path}^{\leq n}(x,X) = \mathrm{path}^{\leq\infty}(x,X) \wedge \neg\, \underline{\exists}\, x_0 \ldots \underline{\exists}\, x_n. \\ (x_0 \in X \wedge \cdots \wedge x_n \in X \wedge x < x_0 < x_1 < \cdots < x_n)$$

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$\xi(x,X)$ computes the probability of this finite path.

wCTL **is a fragment of** wMSO **on gPFA**

Proof: Translation of $\mu(\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2)$

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$$\mathrm{path}^{\leq\infty}(x,X) = x \in X$$
$$\wedge \forall z.(z \in X \xrightarrow{+} (z = x \,\underline{\vee}\, \underline{\exists}\, y.(y \in X \wedge y \lessdot z)))$$
$$\wedge \neg \underline{\exists}\, y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z')$$

if $n \in \mathbb{N}$, $$\mathrm{path}^{\leq n}(x,X) = \mathrm{path}^{\leq\infty}(x,X) \wedge \neg \underline{\exists}\, x_0 \ldots \underline{\exists}\, x_n.$$
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$\mathrm{path}^{\leq n}(x,X) \wedge \underline{\psi}(x,X)$ is a boolean formula which holds if and only if X is a minimal path satisfying $\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2$.

$\xi(x,X)$ computes the probability of this finite path.

$\exists X$ computes the sum of the probability of such paths.

Plan

Weighted Automata

Weighted MSO Logic

Weighted CTL^* **and** PCTL^*

Weighted CTL^* **versus weighted** MSO

5 Conclusion and Open problems

Conclusion

- There is a very rich theory for probabilistic systems.
  - Various logics for specification
  - Efficient algorithms for model checking
  - and much more (probabilistic bisimulation, . . .)
- Analysis of other quantitative properties is more and more important.
  Reliability, energy consumption, . . .
- We should develop a strong theory for analysis of various quantitative aspects

  Building upon existing theory of weighted automata
  and the large experience in analysing probabilistic systems.

Open problems

Problems on wMSO

- Identify fragments for which satisfiability and model checking are decidable.
- Compare expressivity of wCTL* (or PCTL*) and wMSO on GPFA.
- Compare expressivity of wCTL* (or PCTL*) and wMSO on RPFA.
- Extend the comparison to other semirings.
  E.g. the Expectation semiring — Eisner '01
  Useful to compute expected rewards.
- Find a weighted μ-calculus which contains wCTL and compare its expressivity with wMSO.
  Weighted μ-calculus on words — Meinecke, DLT'09
  Weighted μ-calculus for quantitative games — Fischer, Grädel & Kaiser '08

Open problems

Quantitative bisimulation

- Probabilistic bisimulation — Larsen & Skou, '91
  It is not quantitative, it defines a boolean relation on states.

Open problems

Quantitative bisimulation

- Probabilistic bisimulation — Larsen & Skou, '91
  It is not quantitative, it defines a boolean relation on states.
- Generalized to weighted automata and CTL$ — Buchholz & Kemper '09
  But still not quantitative.

Open problems

Quantitative bisimulation

- Probabilistic bisimulation — Larsen & Skou, '91
  It is not quantitative, it defines a boolean relation on states.
- Generalized to weighted automata and CTL$ — Buchholz & Kemper '09
  But still not quantitative.
- We need to study bisimulation distances expressing how close two states are.
  See Fahrenberg, Larsen & Thrane '09