Thiagarajan's Conjecture

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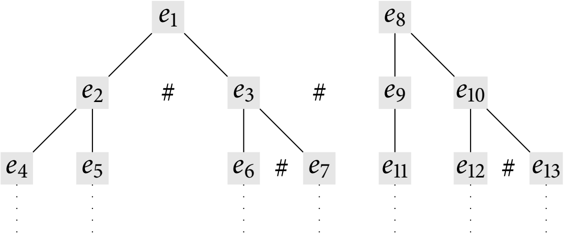
Definition

An event structure ES is a tuple $ES = (E, \leq, \#)$ where $\leq \subset E \times E$ is a partial order called the causality relation and $\# \subset E \times E$ is the conflict relation which is inherited. That is, for $e_1, e_2, e_3 \in E,\ e_1 \# e_2 \lessdot e_3 \Rightarrow e_1 \# e_3$.

An event structure:



Definition

Two events $e_1, e_2 \in E$ are said to be in minimal conflict denoted $e_1 \#_\mu e_2$ if for any events $e'_1, e'_2 \in E$, $e'_1 \leq e_1, e'_1 \# e_2 \Rightarrow e'_1 = e_1$ and $e'_2 \leq e_2, e'_2 \# e_1 \Rightarrow e'_2 = e_2$.

Definition

A configuration is a subset $c \subset E$ such that c is prefix-closed and for every $e_1, e_2 \in c$, $\neg(e_1 \# e_2)$. C_{ES} denotes the set of configurations of ES.

Definition

For a configuration c, let $\#(c)$ denote the set of events that are in conflict with the events and c and $\#_\mu(c)$ denote those in minimal conflict.

Definition

An event e is enabled at a configuration c if $e \notin c$ and $c \cup \{e\}$ is also a configuration. The resulting configuration is denoted by $c \xrightarrow{e}$. An event structure is boundedly enabled if there exists a bound b such that at every configuration, the number of events enabled is at most b.

Definition

- The residue of a configuration c is the set $E \setminus (c \cup \#(c))$.
- c and c' are said to be right invariant, $cR_{ES}c'$ if their residues are isomorphic.
- Given two residues in an R_{ES} class r, I^r_{ES} denotes the restriction of the isomorphism to their minimal events.

Definition

- An event structure is recognisable if it has finitely many R_{ES} equivalence classes.
- An event structure is regular if it is recognisable and boundedly enabled.

Definition

A Σ-labelled net consists of a tuple $N = (P, T, \ell, pre, post, m_0)$ of disjoint finite sets P of places and T of transitions, which are labelled, $\ell : T \to \Sigma$, with two functions $pre, post : T \to 2^P$ specifying the pre and postconditions of a transitions and an initial marking $m_0 \subset P$. A net is 1-safe if all reachable markings are sets.

Theorem (Thiagarajan)

The unfoldings of 1-safe nets are regular trace event structures.

Definition

A net N is called a folding of an event structure ES if the unfolding of N is isomorphic to ES.

Conjecture (Thiagarajan)

Every regular event structure has a 1-safe folding.

Definition

A (Mazurkiewicz) trace alphabet is a pair $M = (\Sigma, I)$ where Σ is a finite non-empty set and $I \subset \Sigma \times \Sigma$ is an irreflexive and symmetric relation called the independence relation.

Definition

Let $M = (\Sigma, I)$ be a trace alphabet. An *M*-labelled event structure $LES = (ES, \lambda)$ where $ES = (E, \leq, \#)$ is an event structure and $\lambda : E \to \Sigma$ is a labelling function which satisfies:

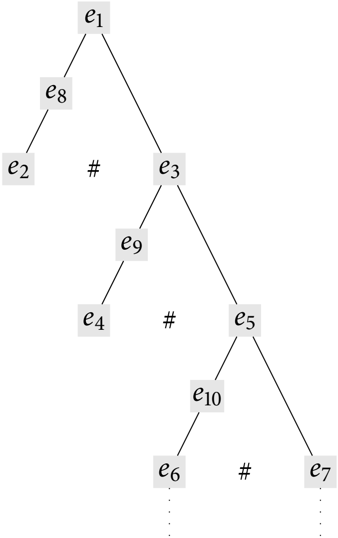
- *LES1* $e \#_\mu e'$ implies $\lambda(e) \neq \lambda(e')$.
- *LES2* $e \lessdot e'$ or $e \#_\mu e'$ implies $(\lambda(e), \lambda(e')) \in D$.
- *LES3* $e \, co \, e'$ implies $(\lambda(e), \lambda(e')) \in I$.

Definition

ES is a trace event structure if and only if there exists a trace alphabet M and an M-labelled event structure LES such that ES is isomorphic to the underlying event structure of LES.

Conjecture (Thiagarajan)

Every regular event structure is also a regular trace event structure.


e_1
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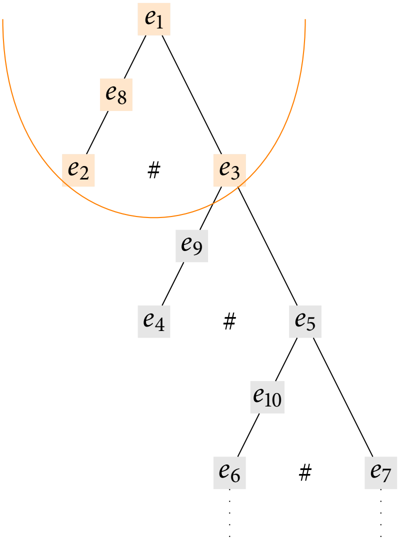
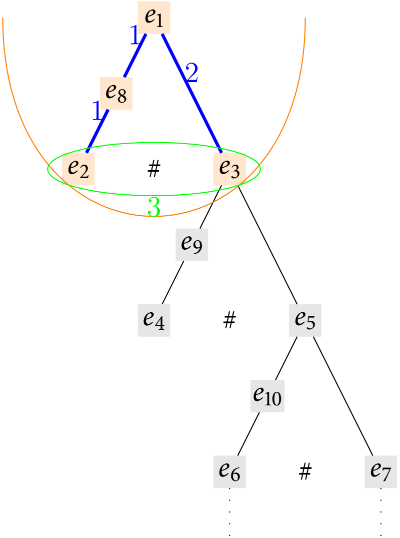
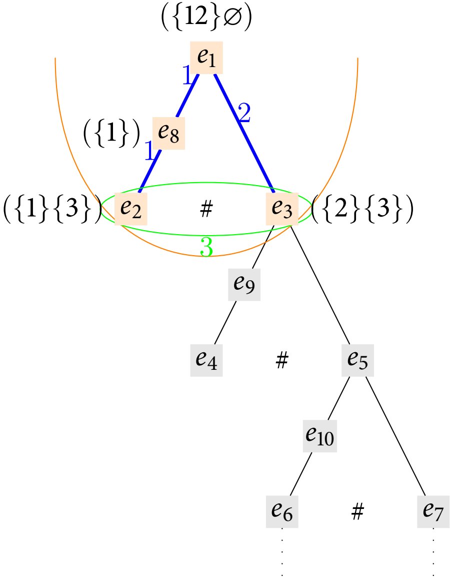
Example



Example



Example



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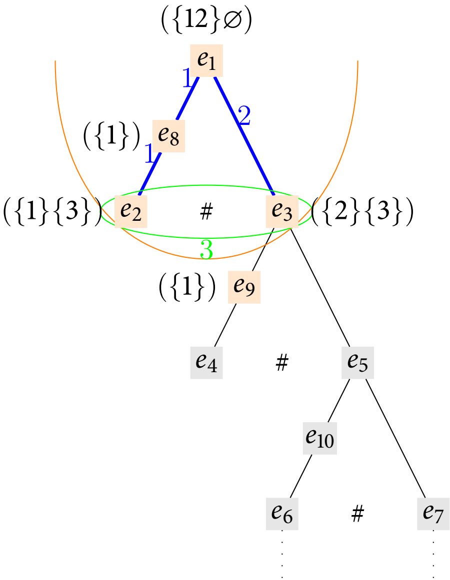

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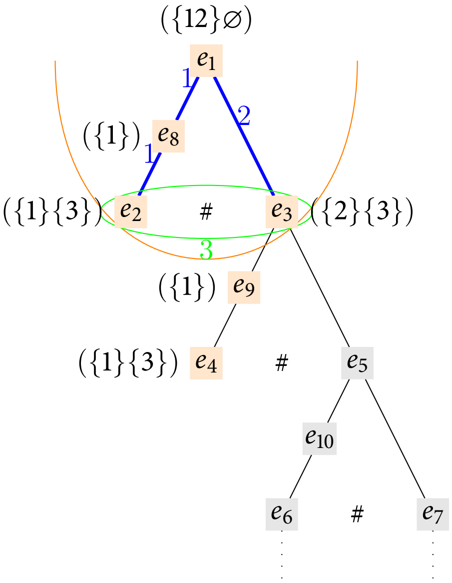

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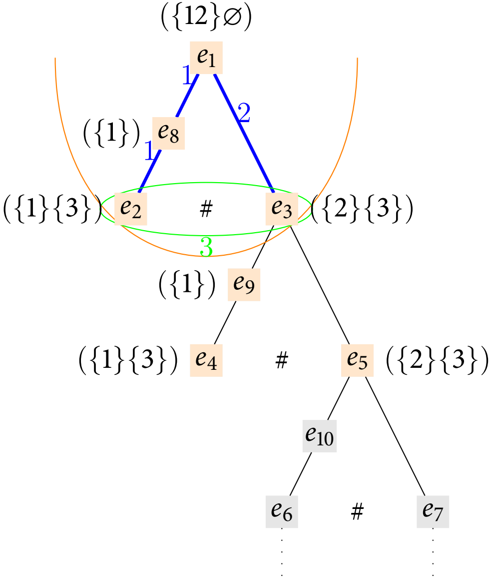

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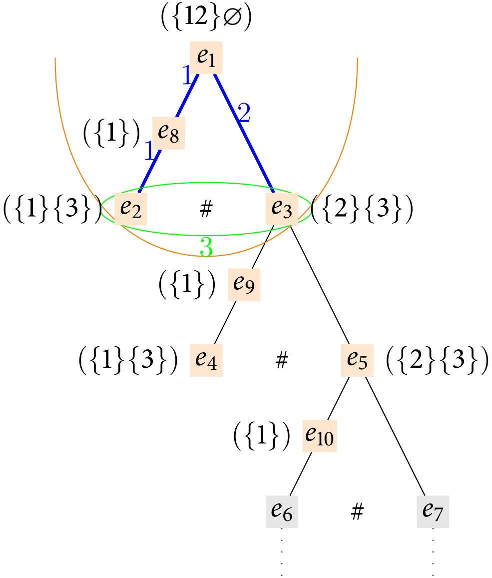

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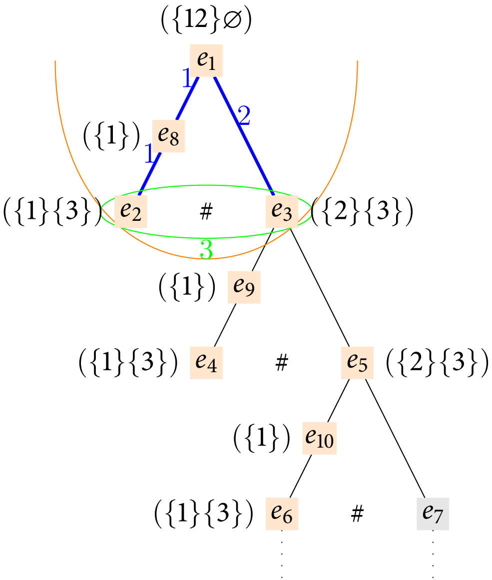
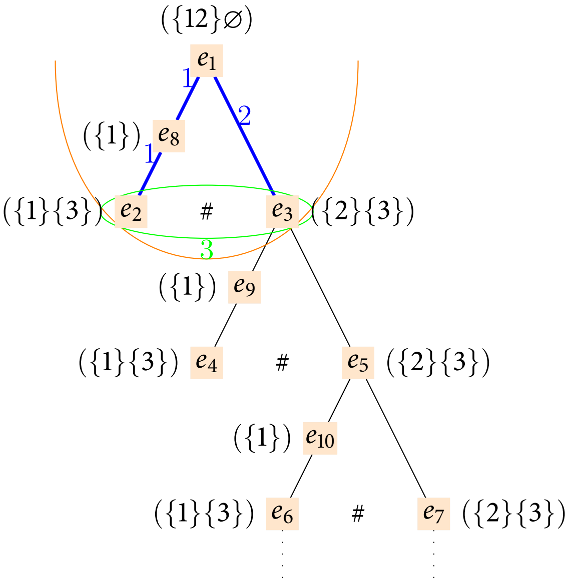
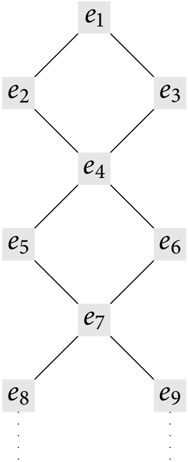


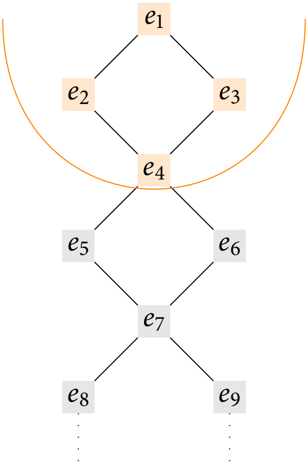
Example



Example[2]



Example[2]



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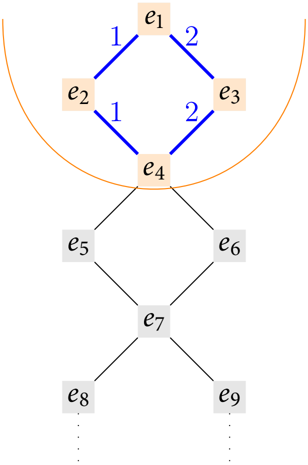

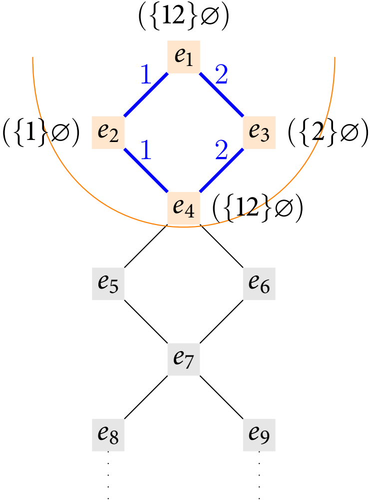

({12}∅)
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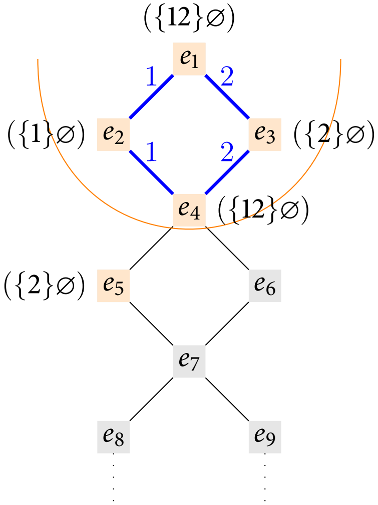
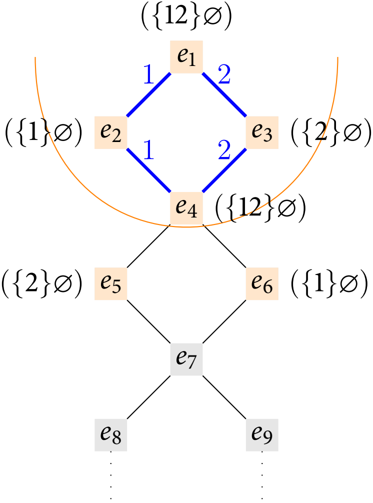
Example[2]





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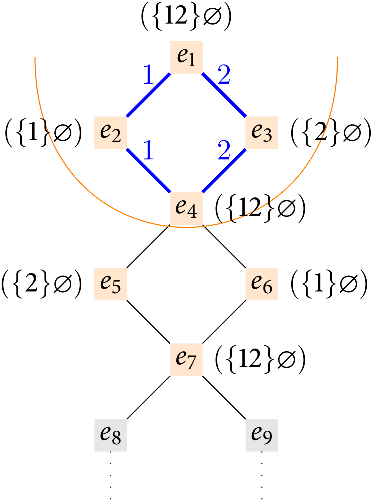

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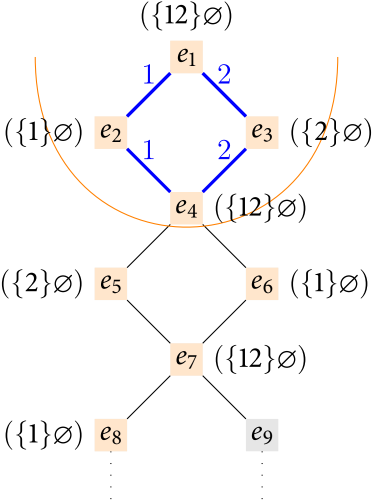

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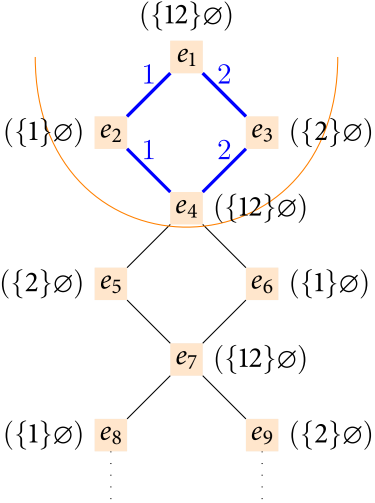

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({12}∅)
({1}∅)
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e9
({2}∅)

Let $E_{fin} = \{e \mid \forall c' \subset c \xrightarrow{e'}, e' \leq e \Rightarrow \neg(c' R_{ES} c)\}$

- If $e_1 < e_2$ and e_2 satisfies the condition above, so does e_1. Hence, E_{fin} is prefix-closed.
- Take the closure of E_{fin} under one step of $\lessdot$ and called the restriction of ES to these events ES_{fin}. ES_{fin} remains prefix closed.
- Let the index of recognisability be n. Along any $<$-path, after at most n events, a configuration containing the latest event with residue isomorphic to one seen earlier will be reached. Thus ES_{fin} has atmost n events along any of its path.
- Since ES is boundedly enabled, ES_{fin} is finite.

Proof[2]

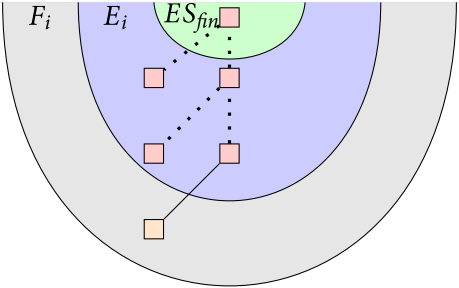
- Let lines be a minimal set of maximal <-chains that cover ES_{fin}.
- Let cliques be the set of maximal $\#_\mu$ cliques of ES_{fin}.
- Put $\Sigma = 2^{lines} \times 2^{cliques}$.
- Put $((a,b),(c,d)) \in I$ if and only if $a \cap c = \varnothing$ and $b \cap d = \varnothing$. Thus $((a,b),(c,d)) \in D$ if and only if $a \cap c \neq \varnothing$ or $b \cap d \neq \varnothing$.
- For every $e \in ES_{fin}$, put $\lambda(e) = (\{i \mid e$ lies on a chain $i\}, \{j \mid e$ lies on a clique $j\})$.

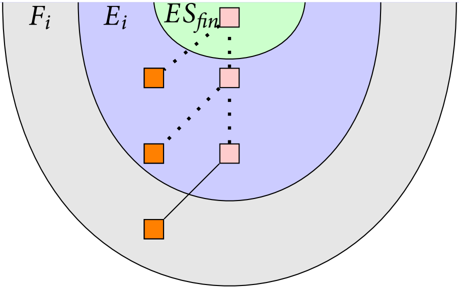
For $e \in ES \setminus ES_{fin}$:

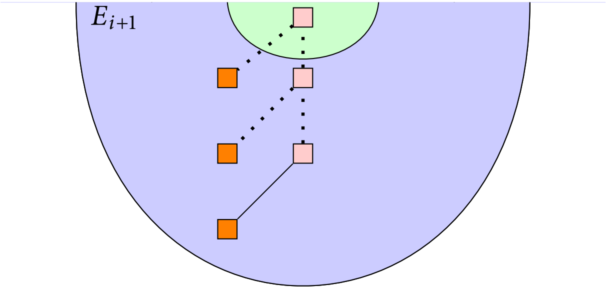
- Order the finite set of residues.
- Suppose E_i has already been labelled.
- Let F_i be the set of events that are enabled by some configuration in E_i.
- For an event $e \in F_i$, consider the predecessors of e till along every such path an event e' is reached such that the minimal configuration c enabling that event has the same residue as another configuration $c_0 \subset c, c_0 \in E_{i-1}$.

- Let r be the minumum of these residues according to the ordering and c be a configuration with residue r. Let p be the path from c to e excluding e.
- For every event e' in p let $c' \supset c$ be the minimum (w.r.t size and residues) configuration enabling it.
- Let c'' be such a configuration enabling e.
- c is called the reference configuration and c'', the base configuration of e.
- Let p_0 be the path of c_0 corresponding to p and $c"_0$ be the configuration corresponding to c''.
- $r(c'') = r(c"_0) = r'$ (say).

- Let e' be the child of p' that is $I^{r'}_{ES}$ equivalent to e.
- Label e with that of e'.
- Let $E_{i+1} = E_i \cup F_i$.
- The final labelled event structure $ES = \bigcup_{i \geq 0} E_i$.


F_i
E_i
ES_{fin}


F_i
E_i
ES_{fin}


E_{i+1}

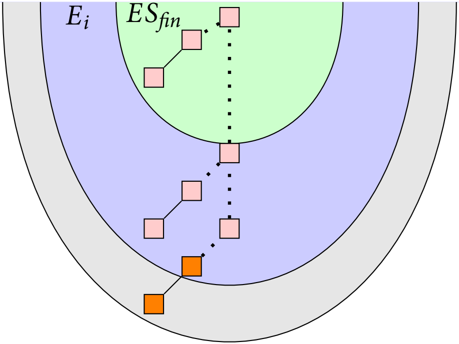
Definition

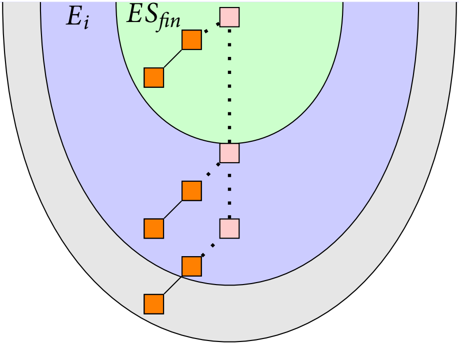
LES2a $e \lessdot e'$ implies $(\lambda(e), \lambda(e')) \in D$.

- If $e \lessdot e'$ and both of them are in ES_{fin} then they share a line. So $(\lambda(e), \lambda(e')) \in D$.
- Otherwise, suppose e' is labelled at the ith iteration and e at the jth iteration such that $i > j$.
- Let the reference configuration and base configuration of e' be c and c' respectively. Note that p may not contain e.
- By induction, we can find $c'_0 \subset ES_{fin}$ such that $r(c') = r(c'_0)$.

Proof[9]

- By our procedure, e' gets the same label as the e'_0 enabled by c'_0 which is $I_{ES}^{r(c')}$ equivalent to e'.
- Since c is the minimum configuration that enables e' and $e \in c$, the minimum configuration that enables e is a subset of c.
- Thus e had been labelled with the label of e_0 such that $e_0 \lessdot e'_0$.
- If e and e' violate LES2a then so do e_0 and e'_0.


E_i
ES_{fin}

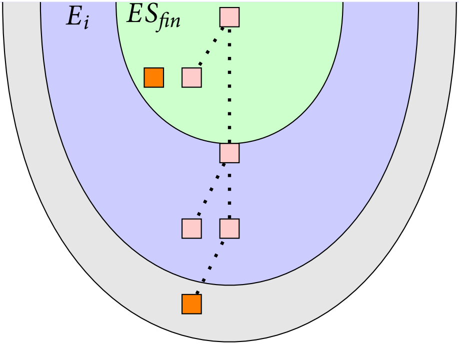

E_i
ES_{fin}

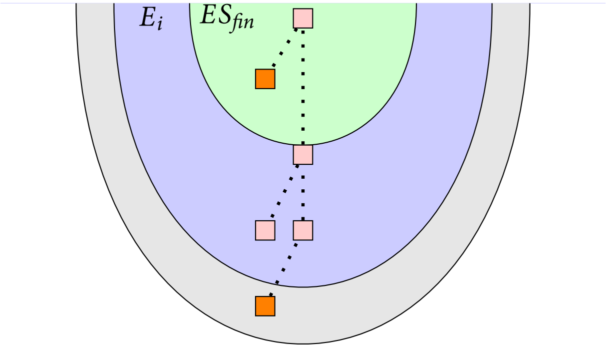
Definition

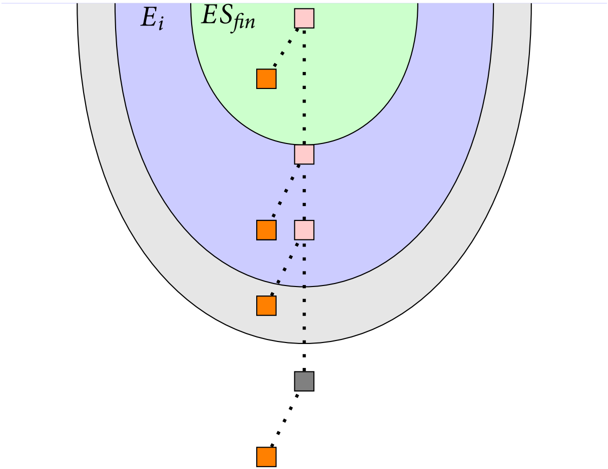
LES3 $e \text{co} e'$ implies $(\lambda(e), \lambda(e')) \in I$.

- If $e \text{co} e'$ and both of them are in ES_{fin} then they do not share a line nor a $\#_\mu$-clique. Hence $(\lambda(e), \lambda(e')) \in I$.
- Otherwise, suppose $e' \in ES_{fin}$ and $e \notin ES_{fin}$.
- Let c and c' be respectively the reference and base configurations of e.
- By induction we can find a configuration c'_0 in ES_{fin} such that $r(c') = r(c'_0)$.

- e was given the same label as the event e_0 enabled at c'_0 that is $I^{r(c')}_{ES}$ equivalent to e.
- If $e_0 \text{co} e'$ then if e and e' violate LES3, then so do e_0 and e'.
- Otherwise the only case is that $e' = e_0$. But then there exists an infinite antichain $e', e, \ldots$ of events contradicting the bounded enabling of ES.


E_i
ES_{fin}


E_i
ES_{fin}


E_i
ES_{fin}

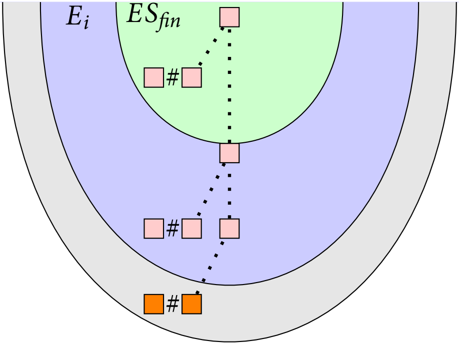
Definition

LES1 $e \#_\mu e'$ implies $\lambda(e) \neq \lambda(e')$.

LES2b $e \#_\mu e'$ implies $(\lambda(e), \lambda(e')) \in D$.

- If $e \#_\mu e'$ and both of them are in ES_{fin}, they share a $\#_\mu$-clique and there are atleast two lines distinguishing them. Hence $\lambda(e) \neq \lambda(e')$ and $(\lambda(e), \lambda(e')) \in D$.
- Otherwise, suppose c and c' are respectively the reference and base configurations of e. By induction we find a configuration c_0' in ES_{fin} such that $r(c_0') = r(c')$.

- If c' is the minimal configuration enabling e' as well, then we have events e_0, e_0' enabled by c_0', $e_0 I_{ES}^{r(c')} e, e_0' I_{ES}^{r(c')} e'$ and $e_0 \# e_0'$.
- If e and e' violate LES1, LES2b then so do e_0, e_0'.
- If e' is not enabled by c' (this is the case of asymmetric confusion), then we can again find an infinite sequence of events $e_0', e', \ldots$ and a configuration c'' such that all of them are enabled at c''. But this contradicts the bounded enabling of ES.



E_i
ES_{fin}
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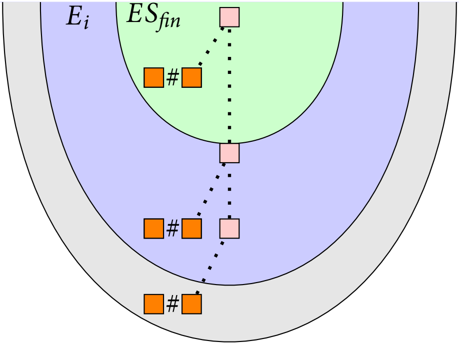
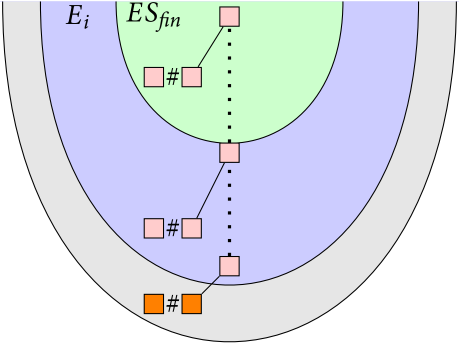
Proof[12]




E_i
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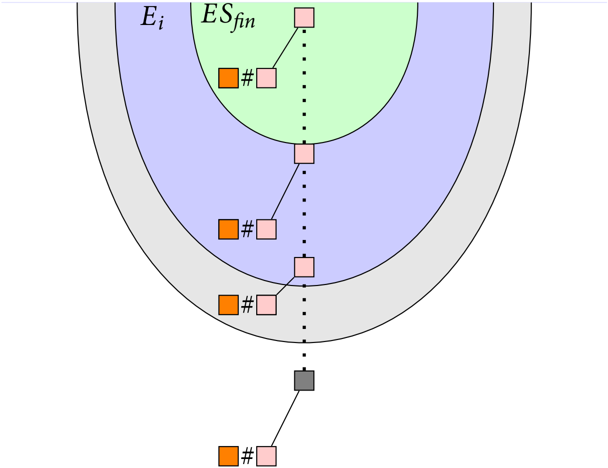
E_i
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