Which local temporal logics are tractable?

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joint work with Paul Gastin

Uniform satisfiability in PSPACE for local temporal logics over Mazurkiewicz traces.
Fundamenta Informaticae, 80:169–197, 2007.
and
Uniform satisfiability problem for local temporal logics over Mazurkiewicz traces.
to appear in *Information and computation*, 2009.

Verification and alternating automata: a success story

Uniform satisfiability problem for LTL

INPUT: LTL-formula φ over finite set of atomic propositions Π
QUESTION: $\exists ? u \in (2^{\Pi})^{\omega}$ with $u \models \varphi$?

Theorem (Sistla & Clarke 1985)

The uniform satisfiability problem for LTL *is* PSPACE-*complete.*

Vardi's proof idea

1. construct alternating automaton $\mathcal{A}_\varphi$ of size $O(|\varphi|)$ s.t. $L(\mathcal{A}_\varphi) = \{u \in (2^{\Pi})^{\omega} \mid u \models \varphi\}$
2. check $L(\mathcal{A}_\varphi)$ for emptiness in space $|\mathcal{A}_\varphi|^{O(1)} = |\varphi|^{O(1)}$ □

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EXTENSION TO TRACES?

Local temporal logics for traces

MSO-definable temporal logics

Polynomial variance and 0-effectiveness

- Variance of Büchi-automata
- … and 0-effectiveness

A temporal logic for words: LTL

$\mathcal{P}$ . . . fixed, countably infinite set of atomic propositions

modalities of LTL

(0) 0-ary: (for $p \in \mathcal{P}$)
   p

(1) unary:
   NOT
   NEXT

(2) binary:
   AND
   UNTIL

LTL-formulas = terms over signature of LTL

A temporal logic for words: LTL

$\mathcal{P}$ ... fixed, countably infinite set of atomic propositions
$\Pi \subseteq \mathcal{P}$ finite, $\Gamma = 2^{\Pi}$
$\Gamma_n^\omega = \Gamma^\omega \times (2^{\mathbb{N}})^n$

modalities of LTL and their semantics

(0) 0-ary: (for $p \in \mathcal{P}$)
$$[\![p]\!]_\Pi = \{(u, \{i\}) \in \Gamma_1^\omega : u = a_0 a_1 \ldots, p \in a_i\}$$

(1) unary:
$$[\![\mathrm{NOT}]\!]_\Pi = \{(u, \{i\}, X) \in \Gamma_2^\omega \mid i \notin X\}$$
$$[\![\mathrm{NEXT}]\!]_\Pi = \{(u, \{i\}, X) \in \Gamma_2^\omega \mid i + 1 \in X\}$$

(2) binary:
$$[\![\mathrm{AND}]\!]_\Pi = \{(u, \{i\}, X, Y) \in \Gamma_3^\omega \mid i \in X \cap Y\}$$
$$[\![\mathrm{UNTIL}]\!]_\Pi \{(u, \{i\}, X, Y) \in \Gamma_3^\omega \mid \exists j \geq i : \begin{array}{l} i, i+1, \ldots, j-1 \in X \\ j \in Y\} \end{array}$$

LTL-formulas = terms over signature of LTL

Traces

Π . . . finite set of process names
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$(A, B) \in D \iff A \cap B \neq \emptyset$

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$\mathbb{R}(\Pi)$ ... set of real traces $(V, \leq, \lambda)$ over Π
$\mathbb{R}_n(\Pi)$ ... set of marked real traces $(V, \leq, \lambda, X_1, \ldots, X_n)$ with $X_1, \ldots, X_n \subseteq V$

A (too) general temporal logic for traces: $\mathcal{L}$

... is given by

- signature Ω (elements: "modality names")
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- signature Ω (elements: "modality names")
  $\mathcal{L}$-formulas = terms over signature Ω of $\mathcal{L}$
- for each finite set of processes Π and $M \in \Omega$ of arity n:

$$[\![M]\!]_\Pi \subseteq \mathbb{R}_{n+1}(\Pi)$$

Example

$$[\![\mathrm{X}]\!]_\Pi = \{(V, \leq, \lambda, \{x\}, Y) \in \mathbb{R}_2(\Pi) \mid \exists y \in Y : x \lessdot y\}$$

$$[\![\mathrm{U}]\!]_\Pi = \{(V, \leq, \lambda, \{x\}, Y, Z) \in \mathbb{R}_3(\Pi) \mid \exists z \in Z : x \leq z \wedge \forall y : (x \leq y \leq z \rightarrow y \in Y)\}$$

A (too) general temporal logic for traces: $\mathcal{L}$

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- signature Ω (elements: "modality names")
  $\mathcal{L}$-formulas = terms over signature Ω of $\mathcal{L}$
- for each finite set of processes Π and $M \in \Omega$ of arity n:

$$[\![M]\!]_\Pi \subseteq \mathbb{R}_{n+1}(\Pi)$$

- then $t, v \models M(\varphi_1, \ldots, \varphi_n) \iff (t, \{v\}, \varphi_1^t, \ldots, \varphi_n^t) \in [\![M]\!]_\Pi$
  for $t = (V, \leq, \lambda) \in \mathbb{R}(\Pi)$, $v \in V$, and $\varphi_i \in \mathcal{L}$ with
  $\varphi_i^t = \{w \in V \mid t, w \models \varphi_i\}$

Example

$$[\![\mathrm{X}]\!]_\Pi = \{(V, \leq, \lambda, \{x\}, Y) \in \mathbb{R}_2(\Pi) \mid \exists y \in Y : x \lessdot y\}$$

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A (too) general temporal logic for traces (continued)

Uniform satisfiability problem for $\mathcal{L}$

INPUT: Π finite set of processes and $\mathcal{L}$-formula φ

QUESTION: $\exists ?(V, \leq, \lambda) \in \mathbb{R}(\Pi)$ and $v \in V$ s.t. $(V, \leq, \lambda), v \models \varphi$?

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Observation

The uniform satisfiability problem is undecidable for "almost all" $\mathcal{L}$

(n-)effective temporal logics

Remark

- elements of $[\![M]\!]_\Pi$ tell that property holds at marked vertex x

(n-)effective temporal logics

Definition

A temporal logic $\mathcal{L}$ is effective if, from Π finite and $M \in \Omega$, one can compute a Büchi-automaton $\mathcal{M}_{\Pi,M}$ ("modality automaton") with

$$(u, X_0, \dots, X_n) \in L(\mathcal{M}_{\Pi,M})$$
$$\Updownarrow$$
$$X_0 = \{x \in \mathbb{N} \mid ([u], \{x\}, X_1, \dots, X_n\}) \in [\![M]\!]_\Pi\}$$

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- elements of $[\![M]\!]_\Pi$ tell that property holds at marked vertex x
- elements of $L(\mathcal{M}_{\Pi,M})$ tell, for each and every vertex, whether property holds or not

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$\mathcal{L}$ is n-effective if $\mathcal{M}_{\Pi,M}$ can be computed in n-fold exponential space ($n = 0$ means polynomial space)

Remark

- elements of $[\![M]\!]_\Pi$ tell that property holds at marked vertex x
- elements of $L(\mathcal{M}_{\Pi,M})$ tell, for each and every vertex, whether property holds or not

The decision procedure

Tractability Theorem

$\mathcal{L}$ effective: uniform satisfiability problem decidable.

$\mathcal{L}$ *n*-effective: uniform satisfiability problem in *n*EXPSPACE.

Proof idea.

Π finite set of processes, $\varphi \in \mathcal{L}$

$\mathrm{Sub}(\varphi)$: set of subformulas of φ

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2. direct product $\mathcal{C}$ of automata $\mathcal{C}_\psi$ accepts $(u, (X_\psi)_{\psi \in \mathrm{Sub}(\varphi)})$ iff $X_\psi = \{x \in \mathbb{N} \mid [u], x \models \psi\}$ for all $\psi \in \mathrm{Sub}(\varphi)$

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3. φ satisfiable in $\mathbb{R}(\Pi)$ iff $\exists(u, (X_\psi)_{\psi \in \mathrm{Sub}(\varphi)}) \in L(\mathcal{C}) : X_\varphi \neq \emptyset$

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complexity: build $\mathcal{C}_\psi$ and $\mathcal{C}$ on-the-fly □

Message

n-effective temporal logics are good!

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But when is a temporal logic n-effective?

Local temporal logics for traces

MSO-definable temporal logics

Polynomial variance and 0-effectiveness

Variance of Büchi-automata

... and 0-effectiveness

Monadic second order logic

atomic formulas:

- $x = y$ (equality in trace $(V, \leq, \lambda)$)
- $x \in X$ (membership)
- $x \lessdot y$ (covering relation in trace $(V, \leq, \lambda)$)
- $p \in \lambda(x)$ for $p \in \mathcal{P}$ (process p is involved in event x)
- $\lambda(x) \subseteq A$ for $A \subseteq \mathcal{P}$ finite (at most the processes from A are involved in event x)

$M\Sigma_n^1$ is set of MSO-formulas of form

$$\exists \overline{Y}_1 \forall \overline{Y}_2 \ldots \exists/\forall \overline{Y}_n : \alpha$$

where α does not contain second-order quantifications.

$M\Sigma_n$-definable temporal logics

Definition

The local temporal logic $\mathcal{L}$ is $M\Sigma_n$-definable if, for every m-ary modality M, there is a formula $\varphi_M \in M\Sigma_n$ for $M \in \Omega$ such that

$$[\![M]\!]_\Pi = \{(t, \{x\}, X_1, \ldots, X_m) \in \mathbb{R}_{m+1} \mid t \models \varphi_M(x, X_1, \ldots, X_m)\}$$

for all finite sets of processes $\Pi \subseteq \mathcal{P}$.

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$$= (\exists U, D \exists z \in X_2 : U = \uparrow x \wedge D = \downarrow z \wedge U \cap D \setminus \{z\} \subseteq X_1) \in M\Sigma_1$$

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(all familiar modalities are $M\Sigma_1$-definable)

Effectiveness of MSO-definable temporal logics

Theorem (Gastin & K '05, '09)

Let $n \geq 0$ be arbitrary.

1. *Every $M\Sigma_n$-definable temporal logic is n-effective.*

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Corollary

The uniform satisfiability problem of all familiar local temporal logics belongs to EXPSPACE ☺

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1. *Every $M\Sigma_n$-definable temporal logic is n-effective.*
2. *Hence: the uniform satisfiability problem of every $M\Sigma_n$-definable temporal logic is in n*EXPSPACE*.*
3. *There exists an $M\Sigma_n$-definable temporal logic whose uniform satisfiability problem is hard for n*EXPSPACE*.*

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3. *There exists an $M\Sigma_n$-definable temporal logic whose uniform satisfiability problem is hard for nEXPSPACE.*

Corollary

The uniform satisfiability problem of all familiar local temporal logics belongs to EXPSPACE ☺, and this approach cannot yield PSPACE ☹.

Local temporal logics for traces

MSO-definable temporal logics

Polynomial variance and 0-effectiveness

- Variance of Büchi-automata
- ... and 0-effectiveness

Is $\mathcal{L}$ 0-effective?

often: automaton $\mathcal{A}_{\Pi,M}$ for $\mathrm{Lin}(\llbracket M \rrbracket_\Pi)$ constructible with $2^{\mathrm{poly}(|\Pi|)}$ states

Is $\mathcal{L}$ 0-effective?

often: automaton $\mathcal{A}_{\Pi,M}$ for $\mathrm{Lin}(\llbracket M \rrbracket_\Pi)$ constructible with $2^{\mathrm{poly}(|\Pi|)}$ states

but: how to transform it into modality automaton $\mathcal{M}_{\Pi,M}$ of polynomial size with

$$(u, X_0, \ldots, X_n) \in L(\mathcal{M}_{\Pi,M})$$
$$\Updownarrow$$
$$X_0 = \{x \in \mathbb{N} \mid (u, \{x\}, X_1, \ldots, X_n\}) \in L(\mathcal{A}_{\Pi,M})\}$$

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Problem description

Problem

given Büchi-automaton $\mathcal{A}$ with $L(\mathcal{A}) \subseteq \Gamma^\omega \times 2^{\mathbb{N}}$
construct Büchi-automaton $\mathcal{M}$ of polynomial size with

$$(u, X) \in L(\mathcal{M}) \\ \Updownarrow \\ X = \{x \in \mathbb{N} \mid (u, \{x\}) \in L(\mathcal{A})\},$$

i.e., with $L(\mathcal{M}) = \{(u, X) \mid \forall x : (x \in X \leftrightarrow (u, \{x\}) \in L(\mathcal{A}))\}$

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i.e., with $L(\mathcal{M}) = \{(u, X) \mid \forall x : (x \in X \leftrightarrow (u, \{x\}) \in L(\mathcal{A}))\}$

Plan

(i) use "General variance" to construct $\overleftarrow{\mathcal{B}}$ for $\{(u, X) \mid \forall x(x \in X \leftarrow (u, \{x\}) \in L(\mathcal{A})\}$

(ii) use "Special variance" to construct $\overrightarrow{\mathcal{B}}$ for $\{(u, X) \mid \forall x(x \in X \rightarrow (u, \{x\}) \in L(\mathcal{A})\}$

General variance

$\mathcal{A} = (Q, I, T, F)$ Büchi-automaton with $L(\mathcal{A}) \subseteq \Gamma^\omega \times 2^{\mathbb{N}}$, $v \in \Gamma^*$

$$GS(v) = \{q \in Q \mid \exists x : I \xrightarrow{(v,\emptyset)} q \text{ or } I \xrightarrow{(v,\{x\})} q\}$$

number of states reachable by (v, X) with $|X| \leq 1$

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Proposition 1

Büchi-automaton $\overleftarrow{\mathcal{B}}$ constructible in space $O(GV(\mathcal{A}) \log |\mathcal{A}|)$ with

$$L(\overleftarrow{\mathcal{B}}) = \{(u, X) \mid \forall x : (x \in X \leftarrow (u, \{x\}) \in L(\mathcal{A}))\}$$

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$$\begin{aligned} L(\overleftarrow{\mathcal{B}}) &= \{(u, X) \mid \forall x : (x \in X \leftarrow (u, \{x\}) \in L(\mathcal{A}))\} \\ &= (\Gamma^\omega \times 2^{\mathbb{N}}) \setminus \{(u, X) \mid \exists x : (x \notin X \wedge (u, \{x\}) \in L(\mathcal{A}))\} \end{aligned}$$

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Proof idea (for finite automata).

reachable states in equivalent deterministic automaton contain at most $m = GV(\mathcal{A})$ original states

$\Rightarrow$ algorithm has to handle m-elements sets of original states □

Special variance

$\mathcal{A} = (Q, I, T, F)$ Büchi-automaton with $L(\mathcal{A}) \subseteq \Gamma^\omega \times 2^{\mathbb{N}}$, $v \in \Gamma^*$, $w \in \Gamma^\omega$

$$SS(v, w) = \left\{ q \in Q \mid \exists x \exists \text{ successful run } : \begin{array}{l} I \xrightarrow{(v,\emptyset)} q \xrightarrow{(w,\{x\})} \text{ or} \\ I \xrightarrow{(v,\{x\})} q \xrightarrow{(w,\emptyset)} \end{array} \right\}$$

set of states reachable after v in successful run on (vw, X) with $|X| \leq 1$

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maximal number of states reachable after v in successful run on (vw, X) with $|X| \leq 1$

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Proposition 2

Büchi-automaton $\overrightarrow{\mathcal{B}}$ constructible in space $O(SV(\mathcal{A}) \log |\mathcal{A}|)$ with

$$L(\overrightarrow{\mathcal{B}}) = \{(u, X) \mid \forall x : (x \in X \rightarrow (u, \{x\}) \in L(\mathcal{A}))\}$$

Special variance

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Special variance

Proposition 2

Büchi-automaton $\overrightarrow{\mathcal{B}}$ constructible in space $O(SV(\mathcal{A})\log|\mathcal{A}|)$ with

$$L(\overrightarrow{\mathcal{B}}) = \{(w, X) \mid \forall x : (x \in X \rightarrow (u, \{x\}) \in L(\mathcal{A}))\}$$

Proof idea

1. construct alternating automaton for $L(\overrightarrow{\mathcal{B}})$ s.t. slices in minimal successful run dags are of form $\{q_1, \ldots, q_n, B\}$ with $q_i \in Q$, $B \subseteq Q$, $n, |B| \leq m = SV(\mathcal{A})$

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   $\Rightarrow$ $\overrightarrow{\mathcal{B}}$ constructible in space $O(m \log(|\mathcal{A}|))$ □

Local temporal logics for traces

MSO-definable temporal logics

Polynomial variance and 0-effectiveness

- Variance of Büchi-automata
- … and 0-effectiveness

0-effectiveness of modalities

Recall

1. n-ary modality is 0-effecitve, if modality-automaton $\mathcal{M}_{M,\Pi}$ that accepts

$$\{(u, X_0, \overline{X}) \mid \forall x : (x \in X_0 \leftrightarrow ([u], \{x\}, \overline{X}) \in [\![M]\!]_\Pi)\}$$

computable in space $\mathrm{poly}(\Pi)$.

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computable in space $\mathrm{poly}(\Pi)$.

2. If all modalities are 0-effective, then the satisfiability problem is in PSPACE.

0-effectiveness of modalities I

Proposition A

M m-ary modality s.t. Büchi-automaton $\mathcal{A}_{M,\Pi}$ is computable from Π in polynomial space with
$L(\mathcal{A}_{M,\Pi}) = \mathrm{Lin}([\![M]\!]_{\Pi}\})$ and $GV(\mathcal{A}_{M,\Pi}) \in \mathrm{poly}(|\Pi|)$.
Then M is 0-effective.

0-effectiveness of modalities I

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Proof.

1. $\overleftarrow{\mathcal{B}}$ from Prop. 1 accepts
$\{(u, X_0, \overline{X}) \mid \forall x : (x \in X_0 \leftarrow ([u], \{x\}, \overline{X}) \in [\![M]\!]_\Pi)\}$

□

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2. $SV(\mathcal{A}) \leq GV(\mathcal{A})$, hence $\overrightarrow{\mathcal{B}}$ from Prop. 2 accepts
   $\{(u, X_0, \overline{X}) \mid \forall x : (x \in X_0 \rightarrow ([u], \{x\}, \overline{X}) \in [\![M]\!]_\Pi)\}$

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then $\mathcal{M}_{M,\Pi}$ for $L(\overrightarrow{\mathcal{B}}) \cap L(\overleftarrow{\mathcal{B}})$ constructible in space
$O(\mathrm{poly}(|\Pi|) \cdot \log(|\mathcal{A}_{M,\Pi}|)) = \mathrm{poly}(|\Pi|)$ □

0-effectiveness of modalities I

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Examples

"strict universal until" SU, "universal until" U, "next" X,
"process-based next" X_p, "process-based until" U_p
"strict universal since" SS, "universal since" S, "yesterday" Y,
"process-based yesterday" Y_p, "process-based since" S_p
path modalities EU, ES, EG

0-effectiveness of modalities II

Proposition B

M m-ary modality s.t. Büchi-automata $\mathcal{A}_{M,\Pi}$ and $\overline{\mathcal{A}_{M,\Pi}}$ are computable from Π in polynomial space with

1. $L(\mathcal{A}_{M,\Pi}) = \mathrm{Lin}(\llbracket M \rrbracket_\Pi\})$ and $SV(\mathcal{A}_{M,\Pi}) \in \mathrm{poly}(|\Pi|)$
2. $L(\overline{\mathcal{A}_{M,\Pi}}) = \mathrm{Lin}(\llbracket M \rrbracket_\Pi\})^{\mathrm{co}}$ and $SV(\overline{\mathcal{A}_{M,\Pi}}) \in \mathrm{poly}(|\Pi|)$

Then M is 0-effective.

Proof

1. $\overrightarrow{\mathcal{B}}$ from Prop. 2 accepts $\{(u, X_0, \overline{X}) \mid \forall x : (x \in X_0 \to ([u], \{x\}, \overline{X}) \in \llbracket M \rrbracket_\Pi)\}$

0-effectiveness of modalities II

Proposition B

M m-ary modality s.t. Büchi-automata $\mathcal{A}_{M,\Pi}$ and $\overline{\mathcal{A}_{M,\Pi}}$ are computable from Π in polynomial space with

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Then M is 0-effective.

Proof

2. $\mathcal{B}$ from Prop. 2 accepts
   $\{(u, Y, \overline{X}) \mid \forall x : (x \in Y \rightarrow ([u], \{x\}, \overline{X}) \notin [\![M]\!]_\Pi)\}$
   modify into $\overleftarrow{\mathcal{B}}$ for
   $\{(u, X_0, \overline{X}) \mid \forall x : (x \notin X_0 \rightarrow ([u], \{x\}, \overline{X}) \notin [\![M]\!]_\Pi)\}$
   which equals
   $\{(u, X_0, \overline{X}) \mid \forall x : (x \in X_0 \leftarrow ([u], \{x\}, \overline{X}) \in [\![M]\!]_\Pi)\}$ □

0-effectiveness of modalities II

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Then M is 0-effective.

Examples

"Thiagarajan's process-based next" $\mathcal{O}_p$, "…until" $\mathcal{U}_p$,
"…since" $\mathcal{S}_p$
"exists concurrently" Eco

Final result

Main theorem

The uniform satisfiability problem for

(1) $M\Sigma^1_n$-definable local temporal logics is nEXPSPACE-complete.

(2) the temporal logic based on all familiar local modalities is PSPACE-complete.

Proof of (2).

(a) all familiar local modalities are 0-effective: corollary to Prop. A and B

(b) result follows from Tractability Theorem □

Final result

Main theorem

The uniform satisfiability problem for

(1) $M\Sigma^1_n$-definable local temporal logics is nEXPSPACE-complete.

(2) the temporal logic based on all familiar local modalities is PSPACE-complete.

analogous statements for the model checking problem hold as well.