Automata for Real-Time Systems

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Let $T\Sigma^*$ denote the set of **all timed words**

Universality: Given A, is $\mathcal{L}(A) = T\Sigma^*$?

Inclusion: Given A, B, is $\mathcal{L}(B) \subseteq \mathcal{L}(A)$?

Universality and inclusion are **undecidable** when A has **two clocks** or more

A theory of timed automata

Alur and Dill. *TCS'94*

Lecture 5:

A decidable case of the inclusion problem

Universality: Given A, is $\mathcal{L}(A) = T\Sigma^*$?

Inclusion: Given A, B, is $\mathcal{L}(B) \subseteq \mathcal{L}(A)$?

One-clock restriction

Universality and inclusion are **decidable** when A has at most **one clock**

On the language inclusion problem for timed automata: Closing a decidability gap

Ouaknine and Worrell. *LICS'05*

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Universality and inclusion are **decidable** when A has at most **one clock**

On the language inclusion problem for timed automata: Closing a decidability gap

Ouaknine and Worrell. *LICS'05*

In this lecture: universality for one clock TA

Step 0:

Well-quasi orders and Higman's Lemma

Quasi-order

Given a set Q, a quasi-order is a **reflexive** and **transitive** relation:

$$\sqsubseteq \quad \subseteq \quad Q \times Q$$

- $(\mathbb{N}, \leq)$
- $(\mathbb{Z}, \leq)$

$$\text{Let } \Lambda = \{A, B, \ldots, Z\}, \quad \Lambda^* = \{\text{set of words}\}$$

- $(\Lambda^*, \text{ lexicographic order } \sqsubseteq_L)$: $AAAB \sqsubseteq_L AAB \sqsubseteq_L AB$
- $(\Lambda^*, \text{ prefix order } \subseteq_P)$: $AB \subseteq_P ABA \subseteq_P ABAA$
- $(\Lambda^*, \text{ subword order } \preccurlyeq)$ $HIGMAN \preccurlyeq HIGHMOUNTAIN$ [OW'05]

Well-quasi-order

An infinite sequence $\langle q_1, q_2, \dots \rangle$ in $(Q, \sqsubseteq)$ is saturating if $\exists\, i < j : q_i \sqsubseteq q_j$

A quasi-order $\sqsubseteq$ is a well-quasi-order (wqo) if **every** infinite sequence is saturating

- $(\mathbb{N}, \leq)$
- $(\mathbb{Z}, \leq)$
- $(\Lambda^*,$ lexicographic order $\sqsubseteq_L)$:
- $(\Lambda^*,$ prefix order $\subseteq_P)$:
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- $(\mathbb{N}, \leq)$ $\surd$
- $(\mathbb{Z}, \leq)$ $\times$ $-1 \geq -2 \geq -3, \dots$
- $(\Lambda^*,$ lexicographic order $\sqsubseteq_L)$: $\times$ $B \sqsupseteq_L AB \sqsupseteq_L AAB \dots$
- $(\Lambda^*,$ prefix order $\subseteq_P)$: $\times$ $B, AB, AAB, \dots$
- $(\Lambda^*,$ subword order $\preccurlyeq)$

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- $(\mathbb{N}, \leq)$ ✓
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- $(\Lambda^*,$ subword order $\preccurlyeq)$?

Higman's lemma

Let $\sqsubseteq$ be a quasi-order on Λ

Define the induced **monotone domination order** $\preccurlyeq$ on Λ^* as follows:

$a_1 \dots a_m \preccurlyeq b_1 \dots b_n$ if there exists a **strictly increasing** function

$$f : \{1, \dots, m\} \mapsto \{1, \dots, n\} \text{ s.t}$$

$$\forall\, 1 \leq i \leq m : \; a_i \sqsubseteq b_{f(i)}$$

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Higman'52

If $\sqsubseteq$ is a wqo on Λ, then the induced monotone domination order $\preccurlyeq$ is a wqo on Λ^*

Subword order

$$\begin{aligned} \Lambda &:= \{A, B, \ldots, Z\} \\ \sqsubseteq &:= x \sqsubseteq y \text{ if } x = y \end{aligned}$$

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$$HIGMAN \preccurlyeq HIGHMOUNTAIN$$

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Induced monotone domination order $\preccurlyeq$ is the subword order

$$HIGMAN \preccurlyeq HIGHMOUNTAIN$$

By Higman's lemma, $\preccurlyeq$ is a wqo too

If we start writing an **infinite sequence** of words, we will **eventually** write down a **superword** of an earlier word in the sequence

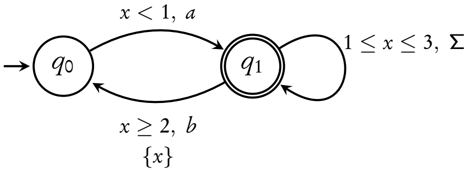
Step 1:

A naive procedure for universality of one-clock TA

Terminology

Let $A = (Q, \Sigma, Q_0, \{x\}, T, F)$ be a timed automaton with one clock

- **Location:** $q_0, q_1, \dots \in Q$
- **State:** (q, u) where $u \in \mathbb{R}_{\geq 0}$ gives value of the clock
- **Configuration:** **finite** set of states



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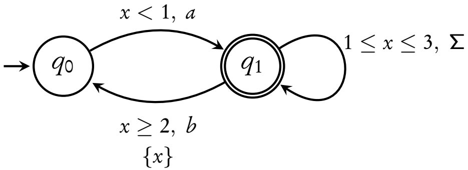
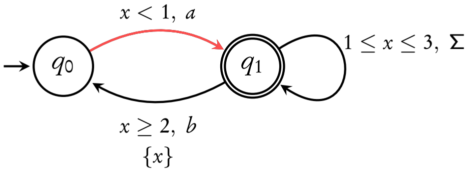
- **Location:** $q_0, q_1, \dots \in Q$
- **State:** (q, u) where $u \in \mathbb{R}_{\geq 0}$ gives value of the clock
- **Configuration:** **finite** set of states $\{(q_1, 2.3), (q_0, 0)\}$



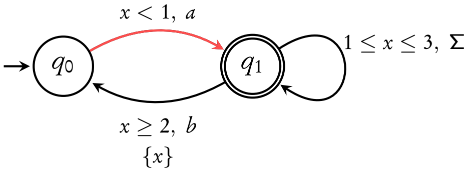
Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a}$$



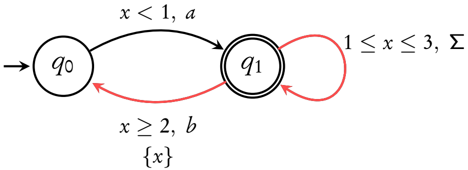
Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\}$$



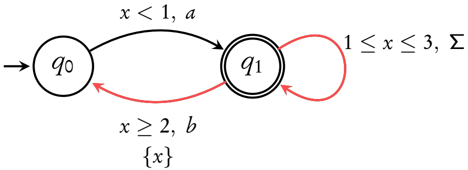
Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\} \xrightarrow{2.1,\ b}$$



Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\} \xrightarrow{2.1,\ b} \{(q_1, 2.3), (q_0, 0)\} \ldots$$



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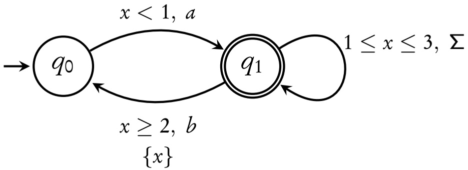
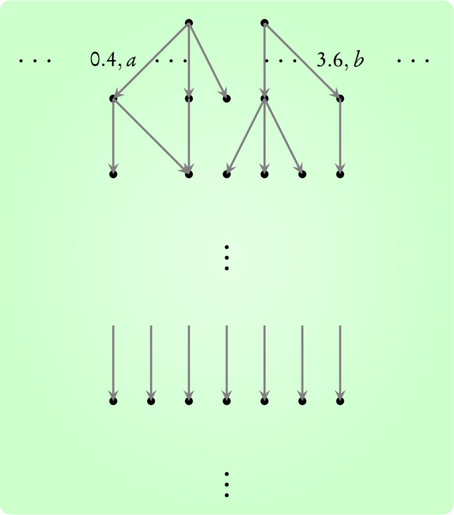


$$C_1 \xrightarrow{\delta,\ a} C_2 \text{ if}$$

$$C_2 = \{\ (q_2, u_2) \mid \exists (q_1, u_1) \in C_1 \text{ s. t. } (q_1, u_1) \xrightarrow{\delta,\ a} (q_2, u_2)\}$$

Labeled transition system of **configurations**



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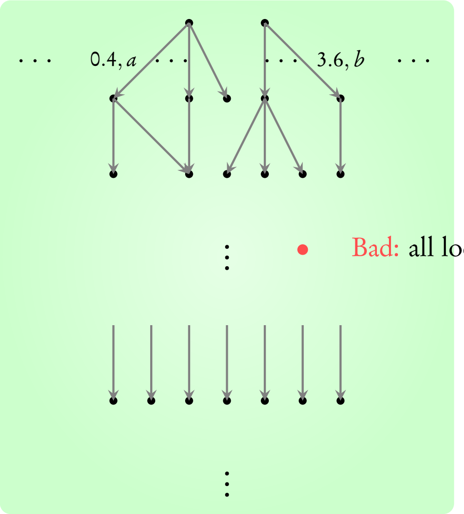
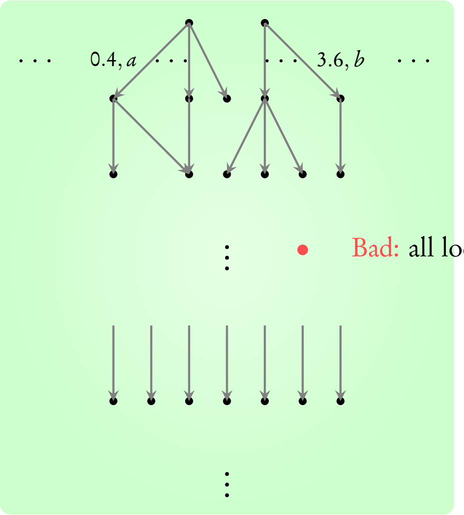


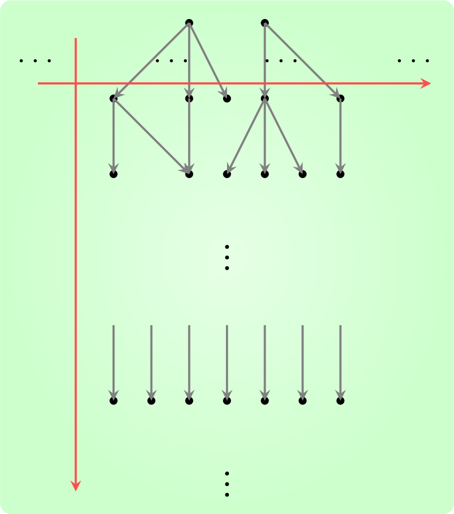
- Bad: all locations **non-accepting**

Labeled transition system of **configurations**

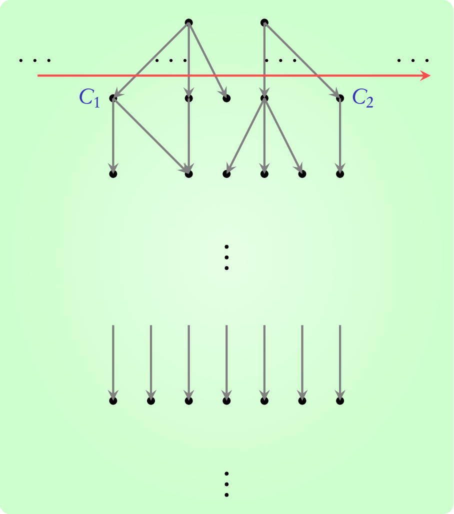


Bad: all locations **non-accepting**

Is a bad configuration **reachable** from some **initial** configuration?



Need to handle **two dimensions** of infinity!

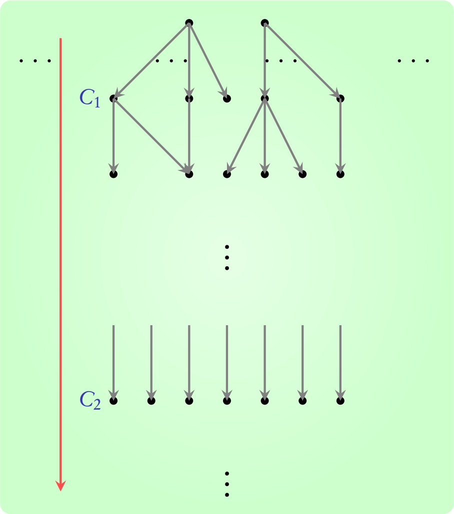


abstraction by **equivalence** $\sim$

$C_1 \sim C_2$ iff:

C_1 goes to a **bad** config. $\Leftrightarrow$ C_2 goes to a **bad** config.

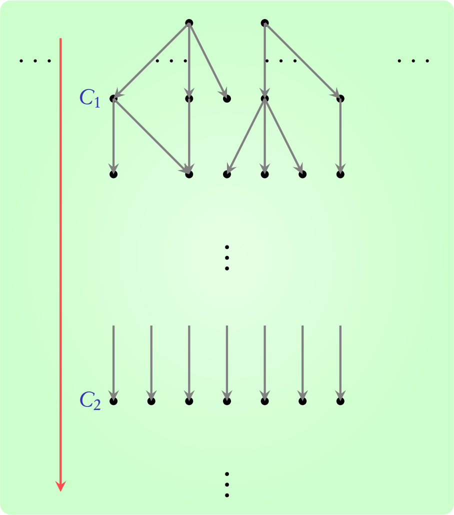
finite **domination** order $\preccurlyeq$



$C_1 \preccurlyeq C_2$ iff:

C_2 goes to a **bad** config $\quad\Rightarrow\quad$ C_1 goes to a **bad** config. too

finite **domination** order $\preccurlyeq$



$C_1 \preccurlyeq C_2$ iff:

C_2 goes to a **bad** config $\quad \Rightarrow \quad$ C_1 goes to a **bad** config. too

No need to explore C_2!

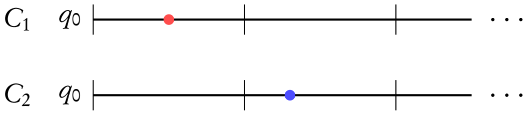
Step 2:

The equivalence

Credits: Examples in this part taken from one of **Ouaknine's talks**

Equivalent configurations: Examples

$$C_1 = \{(q_0, 0.5)\} \not\sim C_2 = \{(q_0, 1.3)\}$$



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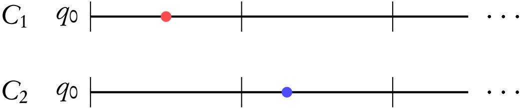
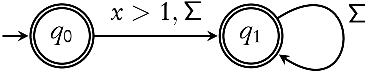
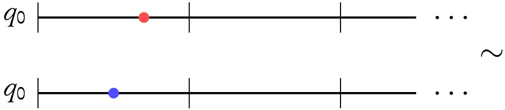
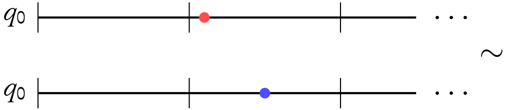




C_2 is universal, but C_1 **rejects** $(a, 0)$


q_0
q_0
$\sim$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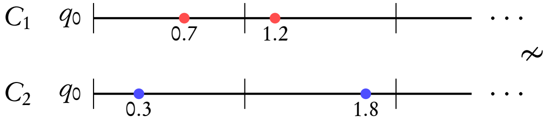
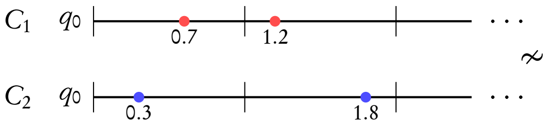
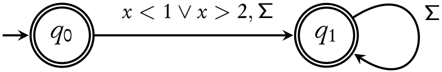
q_0
$\sim$


C_1 q_0
0.7
1.2
$\not\propto$
C_2 q_0
0.3
1.8

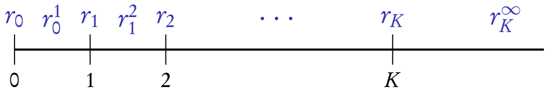




C_2 is universal, but C_1 **rejects** $(a, 0.5)$

Let K be the largest constant appearing in A

Define $REG = \{r_0, r_0^1, r_1, \ldots, r_K, r_K^\infty\}$



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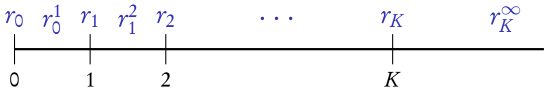

$C = \{(q_1, 0.0), (q_1, 0.3), (q_1, 1.2), (q_2, 1.0), (q_3, 0.8), (q_3, 1.3)\}$

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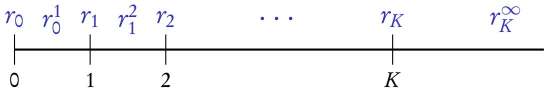

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$$\{(q_1, r_0, 0), (q_1, r_0^1, 0.3), (q_1, r_1^2, 0.2), (q_2, r_1, 0), (q_3, r_0^1, 0.8), (q_3, r_1^2, 0.3)\}$$

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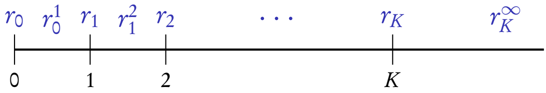

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$$\{(q_1, r_0, 0), (q_2, r_1, 0)\}\ \{(q_1, r_1^2, 0.2)\}\ \{(q_1, r_0^1, 0.3)(q_3, r_1^2, 0.3)\}\ \{(q_3, r_0^1, 0.8)\}$$

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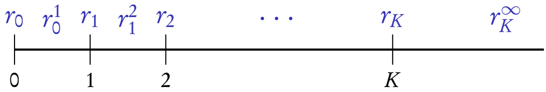

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$$H(C) = \{(q_1, r_0), (q_2, r_1)\} \; \{(q_1, r_1^2)\} \; \{(q_1, r_0^1)(q_3, r_1^2)\} \; \{(q_3, r_0^1)\}$$

Let K be the largest constant appearing in A

$$REG := \{r_0, r_0^1, r_1, \ldots, r_K, r_K^\infty\}$$

$$\Lambda := \mathcal{P}(\ Q \times REG\ )$$

We can give $H : C \mapsto \Lambda^*$ that remembers:

- **integral** part of the clock value (modulo K) in each state of C,
- **order** of **fractional** parts of the clock among different states in C

Equivalence

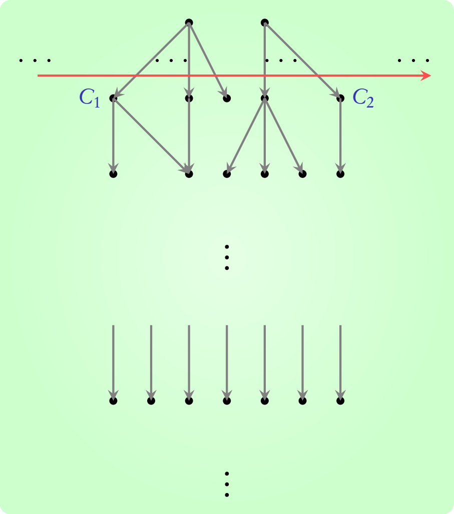
$$C_1 \sim C_2 \quad \text{if} \quad H(C_1) = H(C_2)$$

Equivalence

$C_1 \sim C_2$ if $H(C_1) = H(C_2)$

It can be shown that $\sim$ is a **bisimulation**

C_1 goes to a **bad** config. $\Leftrightarrow$ C_2 goes to a **bad** config.



abstraction by **equivalence** $\sim$

$C_1 \sim C_2$ iff:

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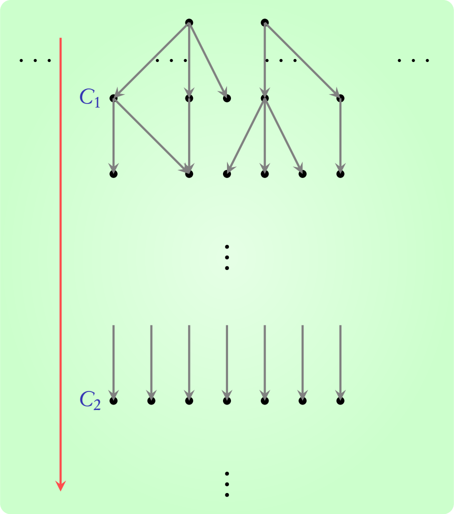
Step 3:

The domination order

finite **domination** order $\preccurlyeq$



$C_1 \preccurlyeq C_2$ iff:

C_2 goes to a **bad** config $\Rightarrow$ C_1 goes to a **bad** config. too

Look at $H(C_1)$ and $H(C_2)$, the words over Λ^*

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Consider the induced monotone domination order $\preccurlyeq$ over Λ^*

$$\{(q_0, r_0)\}\ \ \{(q_1, r_0^1), (q_0, r_2^3)\}$$

$$\preccurlyeq$$

$$\{(q_0, r_0), (q_1, r_1)\}\ \ \{(q_2, r_2^3)\}\ \ \{(q_1, r_0^1), (q_0, r_2^3), (q_2, r_1^2)\}$$

Look at $H(C_1)$ and $H(C_2)$, the words over Λ^*

$$\Lambda = \mathcal{P}(\ Q \times REG\ )$$

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$$\preccurlyeq$$

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Theorem: If $H(C_1) \preccurlyeq H(C_2)$, then $\exists C_2' \subseteq C_2$ s.t. $C_1 \sim C_2$

Look at $H(C_1)$ and $H(C_2)$, the words over Λ^*

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$$\preccurlyeq$$

$$\{(q_0, r_0), (q_1, r_1)\}\ \ \{(q_2, r_2^3)\}\ \ \{(q_1, r_0^1), (q_0, r_2^3), (q_2, r_1^2)\}$$

Theorem: If $H(C_1) \preccurlyeq H(C_2)$, then $\exists C_2' \subseteq C_2$ s.t. $C_1 \sim C_2$

$\subseteq$ is a wqo as Λ is **finite**. Therefore, $\preccurlyeq$ is a **wqo** due to **Higman's lemma**

Final algorithm

- Start from $H(C_0)$, where C_0 is the **initial configuration**
- Successor computation is **effective**
- Termination guaranteed as **domination order is wqo**

A is **universal** iff the algorithm does **not reach a bad node**

One-clock

Universality is **decidable** for one-clock timed automata

One-clock

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For **two clocks**, we know universality is undecidable

One-clock

Universality is **decidable** for one-clock timed automata

For **two clocks**, we know universality is undecidable

Where does this algorithm go wrong when A has two clocks?

Two clocks

State: (q, u, v)

Configuration: $\{(q_1, u_1, v_1), (q_2, u_2, v_2), \ldots, (q_n, u_n, v_n)\}$

At the **least**, the following should be remembered while abstracting:

- relative ordering between fractional parts of x
- relative ordering between fractional parts of y

Current encoding can remember **only one** of them

Other encodings possible?

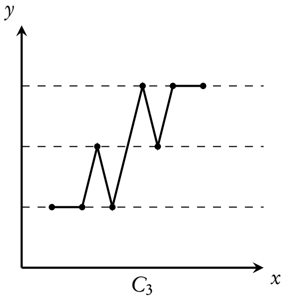
Consider some domination order $\preccurlyeq$

$C_1 \not\preccurlyeq C_2$ if for all $C_2' \subseteq C_2$:

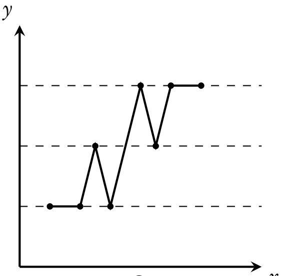
- either relative order of clock x does not match
- or relative order of clock y does not match

In the next slide: **No wqo** possible!

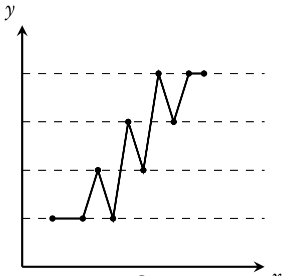
An infinite **non-saturating** sequence $C_1, C_2, C_3, \ldots$



An infinite **non-saturating** sequence $C_1, C_2, C_3, \ldots$

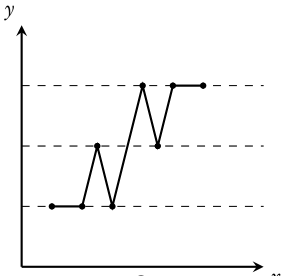


C_3

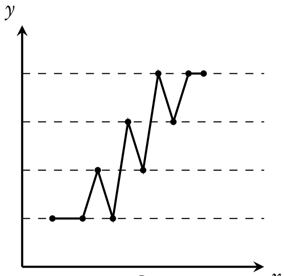


C_4

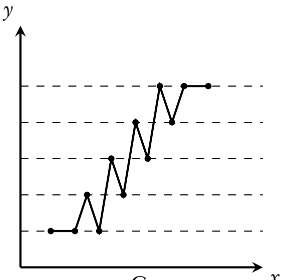
An infinite **non-saturating** sequence $C_1, C_2, C_3, \ldots$



C_3



C_4



C_5

Conclusion

- An algorithm for **universality** when A has one clock
- Can be **extended** for $\mathcal{L}(B) \subseteq \mathcal{L}(A)$ when A has one-clock