Reachability in two clock timed automata is PSPACE-complete

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CMI Seminar

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Reachability in two clock timed automata is PSPACE-complete

[Fearnley, Jurdziǹski, ICALP'13]

Bounded 1-counter automata $\leq_{\mathit{LOGSPACE}}$ **2-clock Timed** automata

[Haase, Ouaknine, Worrell'12]

Subset-sum games $\leq_{PTIME}$ **Bounded 1-counter** automata $\leq_{LOGSPACE}$ **2-clock Timed** automata

[Fearnley, Jurdziǹski'13]

[Haase, Ouaknine, Worrell'12]

Subset-Sum Games

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

$$\psi : \forall \{A_1, B_1\} \; \exists \{E_1, F_1\} \; \ldots \; \forall \{A_n, B_n\} \; \exists \{E_n, F_n\}$$

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Play *P*: 2 players $\forall$, $\exists$

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Play P: 2 players $\forall$, $\exists$

A_1

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

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Play P: 2 players $\forall$, $\exists$

A_1 F_1

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

$$\psi : \forall \{A_1, B_1\} \; \exists \{E_1, F_1\} \; \ldots \; \forall \{A_n, B_n\} \; \exists \{E_n, F_n\}$$

Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\cdots$ B_n E_n

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

$$\psi : \forall \{A_1, B_1\} \exists \{E_1, F_1\} \dots \forall \{A_n, B_n\} \exists \{E_n, F_n\}$$

Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\dots$ B_n E_n

P is **winning** for $\exists$ if $\sum P = T$

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Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\dots$ B_n E_n

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Strategy s for $\exists$:

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Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\dots$ B_n E_n

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Strategy s for $\exists$:

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B_1

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Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\cdots$ B_n E_n

P is **winning** for $\exists$ if $\sum P = T$

Strategy s for $\exists$:

A_1 — (E_1 or F_1)

B_1 — (E_1 or F_1)

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

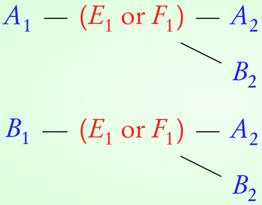
$$\psi : \forall \{A_1, B_1\} \exists \{E_1, F_1\} \ldots \forall \{A_n, B_n\} \exists \{E_n, F_n\}$$

Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\ldots$ B_n E_n

P is **winning** for $\exists$ if $\sum P = T$

Strategy s for $\exists$:



A_1 — (E_1 or F_1) — A_2

B_2

B_1 — (E_1 or F_1) — A_2

B_2

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

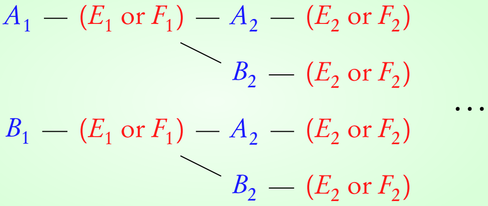
$$\psi : \forall \{A_1, B_1\} \, \exists \{E_1, F_1\} \dots \forall \{A_n, B_n\} \, \exists \{E_n, F_n\}$$

Play P: 2 players $\forall$, $\exists$

A_1 F_1 $\dots$ B_n E_n

P is **winning** for $\exists$ if $\sum P = T$

Strategy s for $\exists$:



A_1 — $(E_1$ or $F_1)$ — A_2 — $(E_2$ or $F_2)$

B_2 — $(E_2$ or $F_2)$

B_1 — $(E_1$ or $F_1)$ — A_2 — $(E_2$ or $F_2)$

B_2 — $(E_2$ or $F_2)$

$\dots$

SSG Instance: (ψ, T) $T, A_i, B_i, E_i, F_i \in \mathbb{N}$

$$\psi : \forall \{A_1, B_1\} \exists \{E_1, F_1\} \ldots \forall \{A_n, B_n\} \exists \{E_n, F_n\}$$

Play P: 2 players $\forall$, $\exists$

$$A_1 \quad F_1 \quad \cdots \quad B_n \quad E_n$$

P is **winning** for $\exists$ if $\sum P = T$

Strategy s for $\exists$:

$$A_1 \text{ — } (E_1 \text{ or } F_1) \text{ — } A_2 \text{ — } (E_2 \text{ or } F_2)$$

$$B_2 \text{ — } (E_2 \text{ or } F_2)$$

$$B_1 \text{ — } (E_1 \text{ or } F_1) \text{ — } A_2 \text{ — } (E_2 \text{ or } F_2)$$

$$B_2 \text{ — } (E_2 \text{ or } F_2)$$

$\cdots$

s is **winning** if all 2^n plays are winning for $\exists$

SSG-Problem: Given (ψ, T) is there a winning strategy for $\exists$

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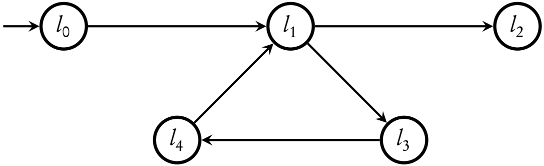
Complexity

SSG-Problem is PSPACE-**complete**

Subset-sum games $\leq_{PTIME}$ **Bounded 1-counter** automata $\leq_{LOGSPACE}$ **2-clock Timed** automata

[Fearnley, Jurdziǹski'13] [Haase, Ouaknine, Worrell'12]

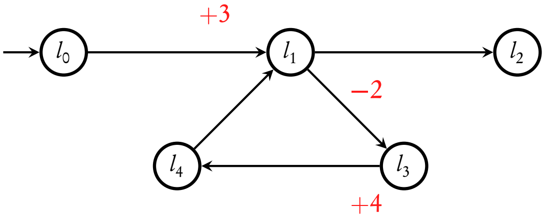
Coming next: **Bounded 1-counter automata**



$\boxed{0}$

c

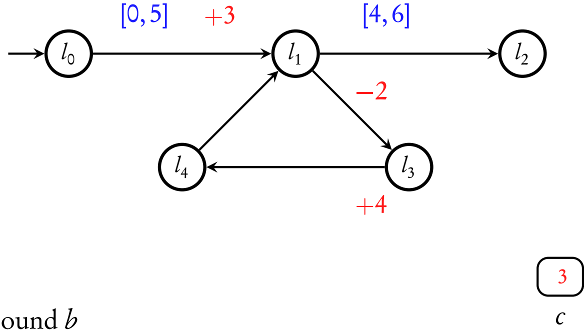
Bound b



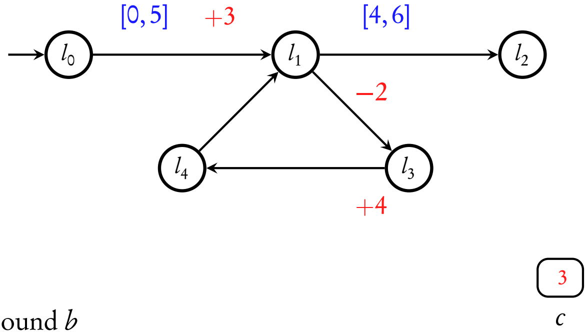
Bound b

$\boxed{3}$

c

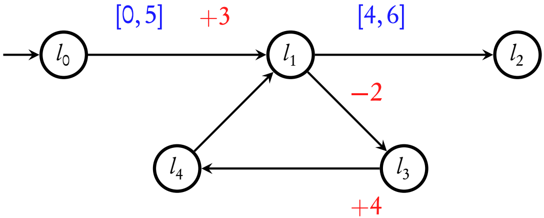


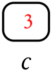
Bound b



Bound b

All guards and updates bounded by b





Bound b

All guards and updates bounded by b

Reachability: Starting at $(l_0, 0)$ can (l_t, c_t) be reached?

Subset-sum games $\leq_{PTIME}$ **Bounded 1-counter** automata $\leq_{LOGSPACE}$ **2-clock Timed** automata

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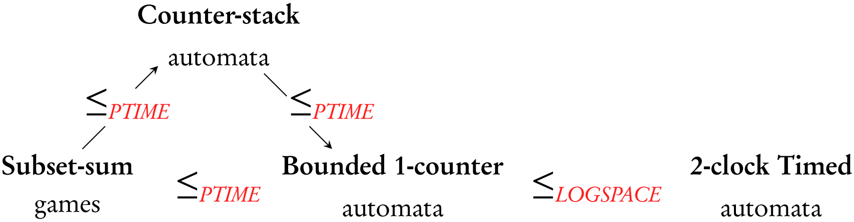


Counter-stack automata

$\leq_{PTIME}$ $\leq_{PTIME}$

Subset-sum games $\leq_{PTIME}$ **Bounded 1-counter** automata $\leq_{LOGSPACE}$ **2-clock Timed** automata

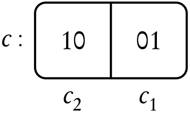
[Fearnley, Jurdziǹski'13] [Haase, Ouaknine, Worrell'12]

$c_1 :$ 0, $\ldots, 3$

$c_2 :$ 0, $\ldots, 3$

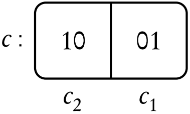
$c_1 : 0, \ldots, 3$

$c_2 : 0, \ldots, 3$



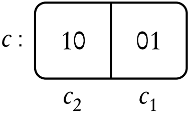
$c_1 : 0, \ldots, 3$

$c_2 : 0, \ldots, 3$



$$c_2 + 1 \longrightarrow c + 3$$

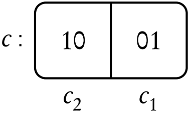
$c_1 :$ 0, $\ldots$ $, 3$
$c_2 :$ 0, $\ldots$ $, 3$



$c :$	10	01
	c_2	c_1

$$c_2 + 1 \longrightarrow c + 3$$
$$c_1 + 1 \longrightarrow c + 1$$

$c_1 : 0, \dots, 3$
$c_2 : 0, \dots, 3$

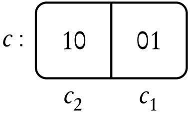


$c_2 + 1 \longrightarrow c + 3$
$c_1 + 1 \longrightarrow c + 1$

$c_2 = 2? \longrightarrow 8 \leq c \leq 11$

$c_1 :$ 0, $\ldots$ $, 3$

$c_2 :$ 0, $\ldots$ $, 3$



$c :$

10

01

c_2

c_1

$c_2 + 1 \quad \longrightarrow \quad c + 3$

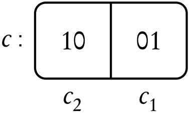
$c_1 + 1 \quad \longrightarrow \quad c + 1$

$c_2 = 2? \quad \longrightarrow \quad 8 \leq c \leq 11$

$c_1 = 2? \quad \longrightarrow \quad c = 2, 6, 10, 14$

$c_1 : 0, \ldots, 3$

$c_2 : 0, \ldots, 3$



$$c_2 + 1 \longrightarrow c + 3 \qquad c_2 = 2? \longrightarrow 8 \leq c \leq 11$$

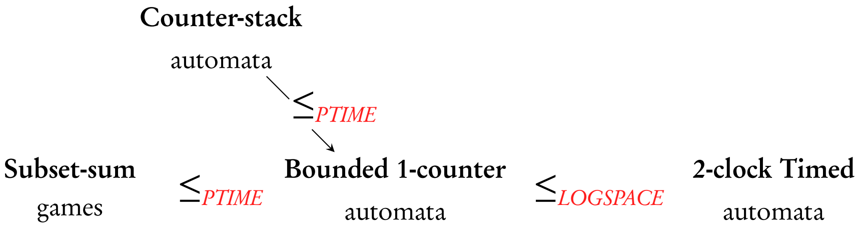
$$c_1 + 1 \longrightarrow c + 1 \qquad c_1 = 2? \longrightarrow c = 2, 6, 10, 14$$

Allowed tests: $c_n = a_n \;\wedge\; c_{n-1} = a_{n-1} \;\wedge\; \ldots \;\wedge\; c_i = a_i$

Counter-stack automata

- **Multiple** counters
- Transitions can:
  - **increment** any counter
  - **test equality**: $c_n = a_n \ \wedge \ c_{n-1} = a_{n-1} \ \wedge \ \dots \ \wedge \ c_i = a_i$
  - **reset** c_i only if $c_i = a_i$ is present
- Each counter is **bounded**

Counter-stack automata $\leq_{PTIME}$ Bounded 1-counter automata



Counter-stack
automata

$\leq_{PTIME}$

Subset-sum
games

$\leq_{PTIME}$

Bounded 1-counter
automata

$\leq_{LOGSPACE}$

2-clock Timed
automata

[Fearnley, Jurdziǹski'13]

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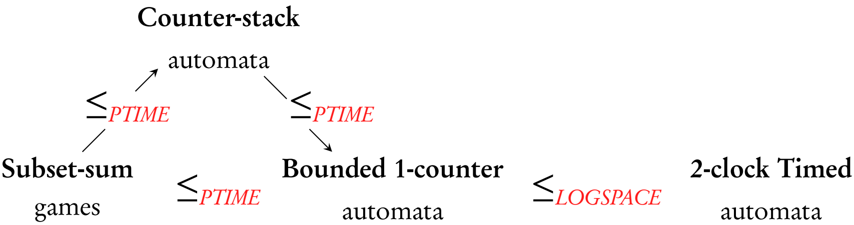

Counter-stack automata

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[Fearnley, Jurdziṅski'13] [Haase, Ouaknine, Worrell'12]

$$(\ \forall\ \{A_1, B_1\}\ \exists\ \{E_1, F_1\}\ \forall\ \{A_2, B_2\}\ \exists\ \{E_2, F_2\},\ T\ )$$

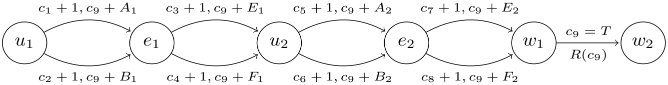


Fig. 1. The play gadget

$$(\forall \{A_1,B_1\} \exists \{E_1,F_1\} \forall \{A_2,B_2\} \exists \{E_2,F_2\}, T)$$

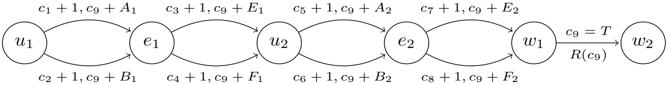


Fig. 1. The play gadget

Play	u_1	e_1	u_2	e_2
1	A_1	E_1 or F_1	A_2	E_2 or F_2
2	A_1	Unchanged	B_2	E_2 or F_2
3	B_1	E_1 or F_1	A_2	E_2 or F_2
4	B_1	Unchanged	B_2	E_2 or F_2

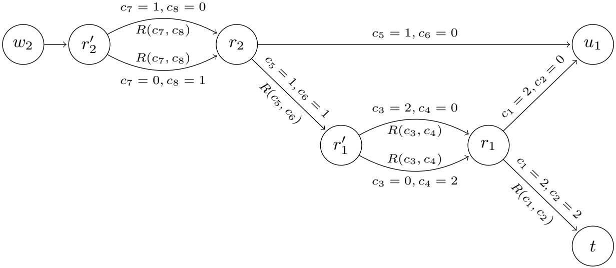


Fig. 2. The reset gadget

Pass	Path
1	$w_2 \to r_2' \to r_2 \to u_1$
2	$w_2 \to r_2' \to r_2 \to r_1' \to r_1 \to u_1$
3	$w_2 \to r_2' \to r_2 \to u_1$
4	$w_2 \to r_2' \to r_2 \to r_1' \to r_1 \to t$

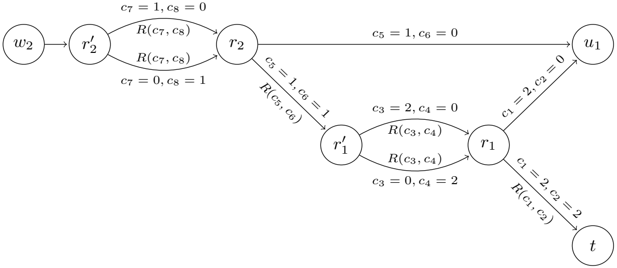
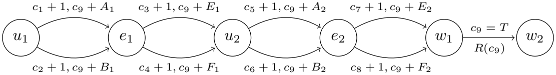


Fig. 2. The reset gadget

Pass	Path
1	$w_2 \to r'_2 \to r_2 \to u_1$
2	$w_2 \to r'_2 \to r_2 \to r'_1 \to r_1 \to u_1$
3	$w_2 \to r'_2 \to r_2 \to u_1$
4	$w_2 \to r'_2 \to r_2 \to r'_1 \to r_1 \to t$

Play	u_1	e_1	u_2	e_2
1	A_1	E_1 or F_1	A_2	E_2 or F_2
2	A_1	Unchanged	B_2	E_2 or F_2
3	B_1	E_1 or F_1	A_2	E_2 or F_2
4	B_1	Unchanged	B_2	E_2 or F_2



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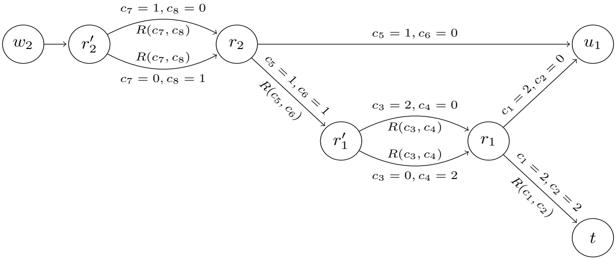


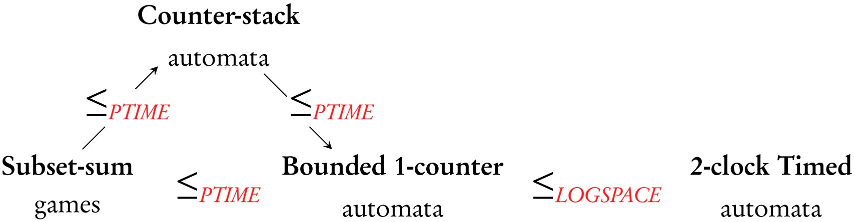
Fig. 2. The reset gadget

t is reached iff there is a winning strategy for ∃

$\leq_{PTIME}$

Given (ψ, T), the counter stack automaton has:

- Locations: $\forall\ i \in [1,n]$: $u_i,\ e_i,\ r_i,\ r'_i,\ w_1,\ w_2,\ t$
- Counters: $k = 2n + 1$
- Bound: $c_k\ \leq\ \Sigma\{A_i, B_i, E_i, F_i\}$ and $c_i \leq 2^n$ for other i
- Transitions: Maximum of two between any two locations



Counter-stack automata

$\leq_{PTIME}$ $\leq_{PTIME}$

Subset-sum games $\leq_{PTIME}$ **Bounded 1-counter** automata $\leq_{LOGSPACE}$ **2-clock Timed** automata

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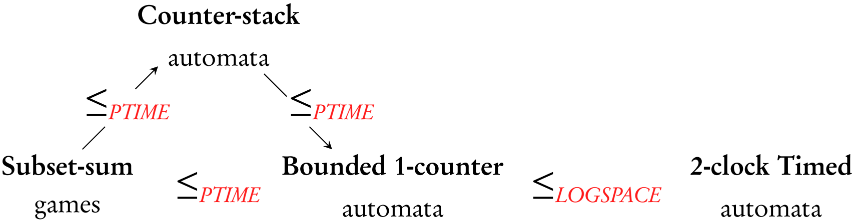
Reachability in two-clock timed automata is PSPACE-hard



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Reachability in two-clock timed automata is PSPACE-hard

Reachability in timed automata is in PSPACE [Alur, Dill '90]

Result

Reachability in 2-clock timed automata is PSPACE-complete