Unit-9: Computation Tree Logic

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NPTEL-course

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Module 2:

CTL*

Recap

- **Path formulae**
  - Express properties of paths
  - LTL
- **Properties on trees**
  - **A** and **E** operators
  - Mixing **A** and **E**

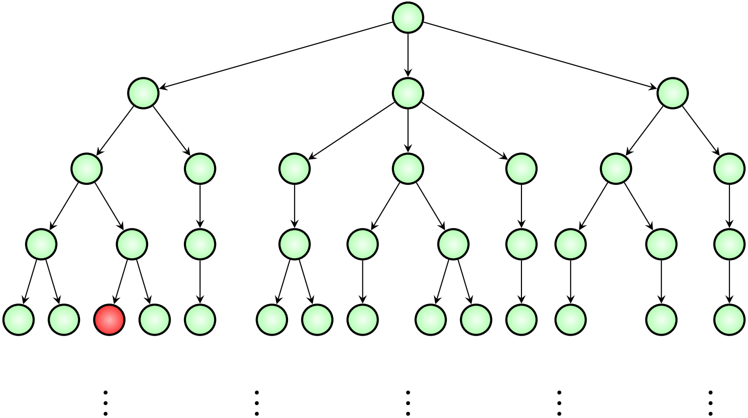
Recap

- **Path formulae**
  - Express properties of paths
  - LTL
- **Properties on trees**
  - **A** and **E** operators
  - Mixing **A** and **E**

Coming next: A **logic** for expressing **properties on trees**

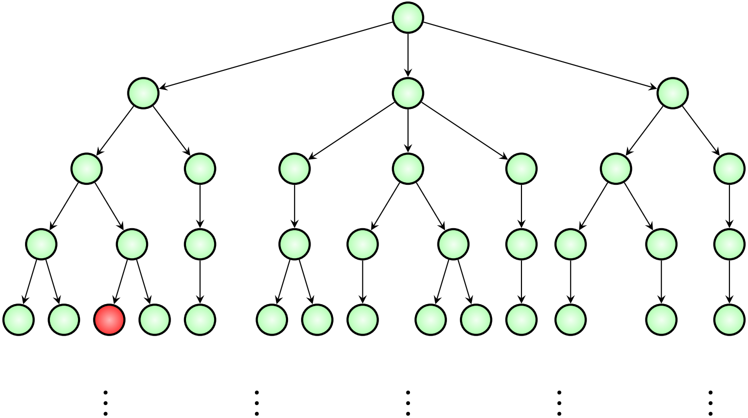
State formulae

$\phi :=$



State formulae

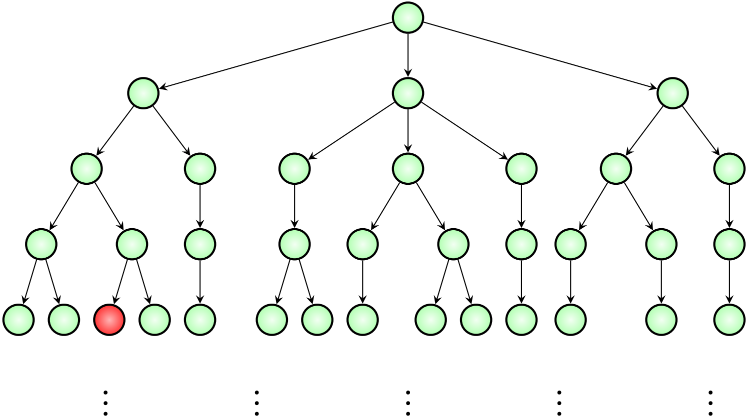
$\phi :=\ \text{true}\ |$



State formulae

$$\phi := \ \text{true} \mid p_i \mid$$

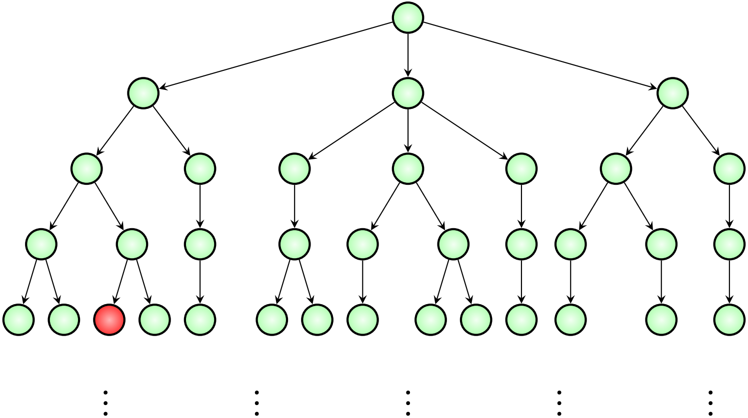
$p_i \in AP$



State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid$$

$p_i \in AP$ $\qquad$ ϕ_1, ϕ_2 : State formulae



State formulae

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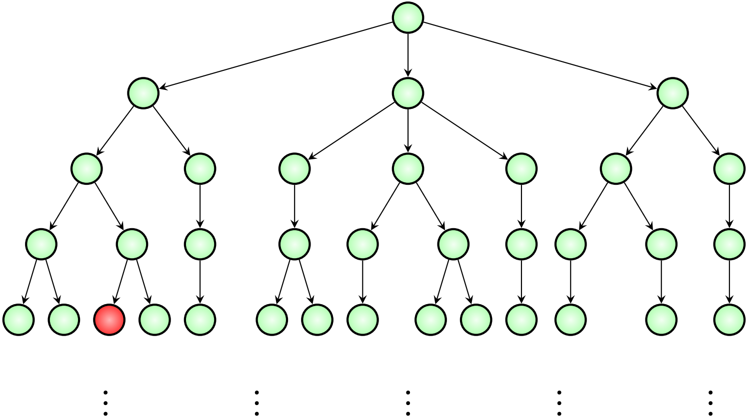

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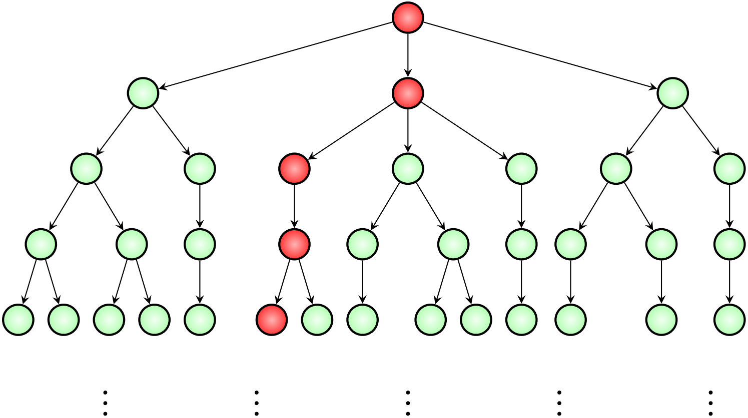

Path formulae

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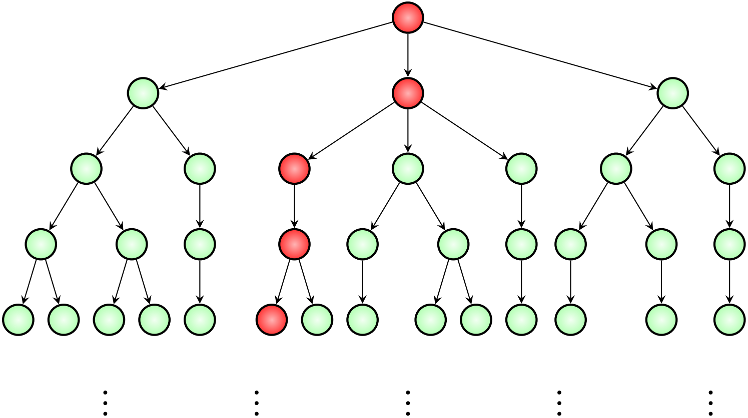
ϕ : State formula



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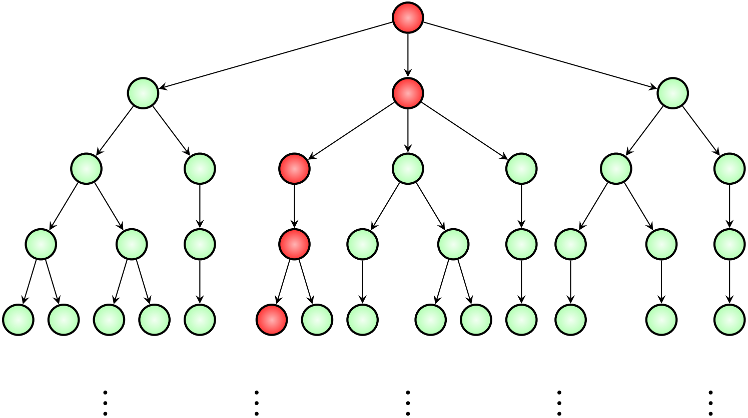

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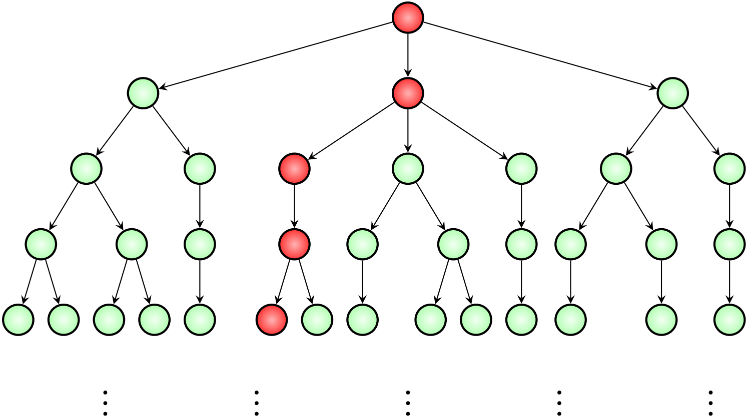

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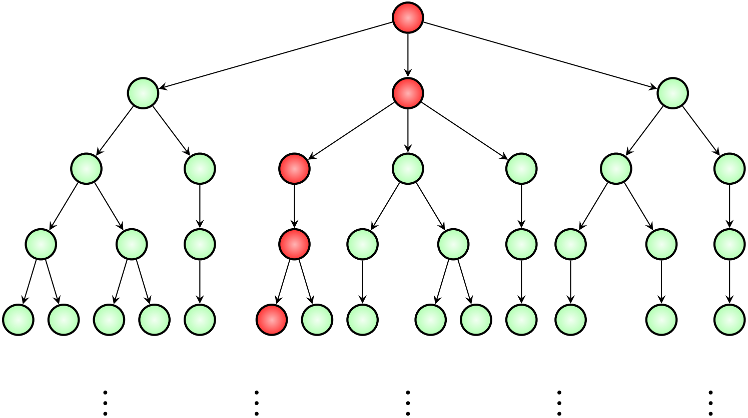

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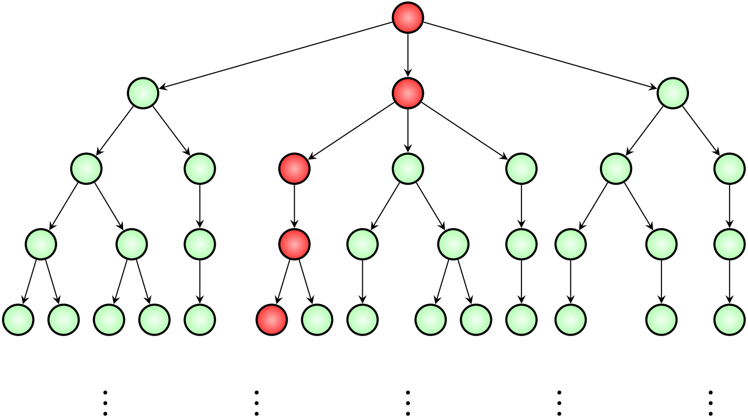

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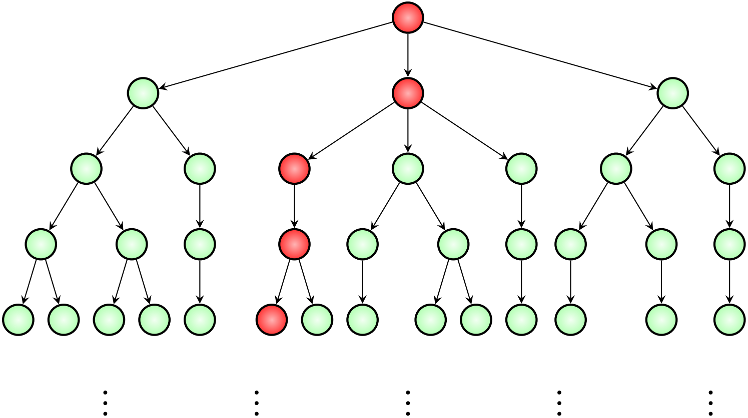

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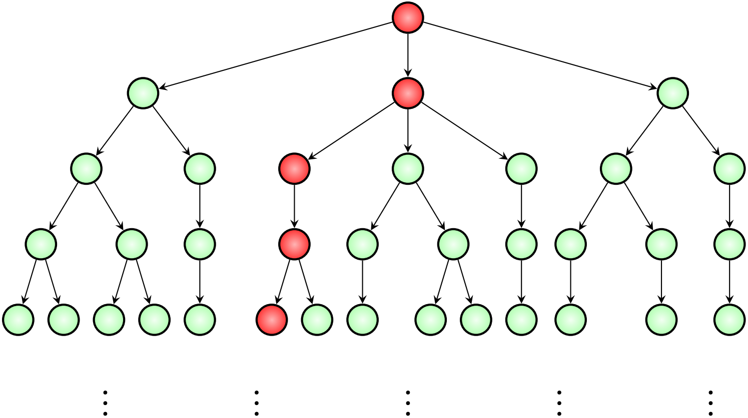

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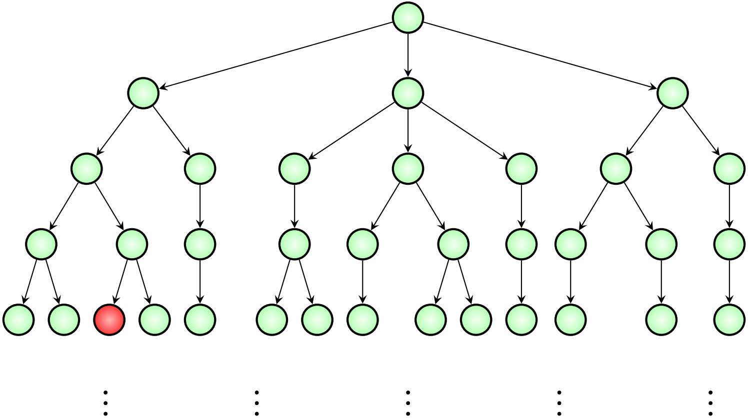
$p_i \in AP$ $\quad$ ϕ_1, ϕ_2 : State formulae



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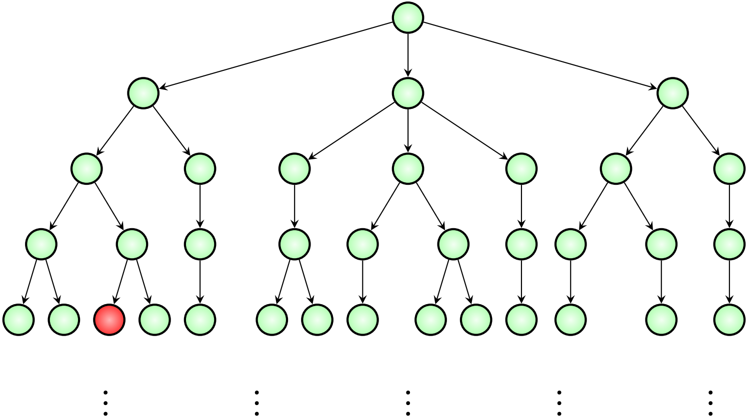
$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula



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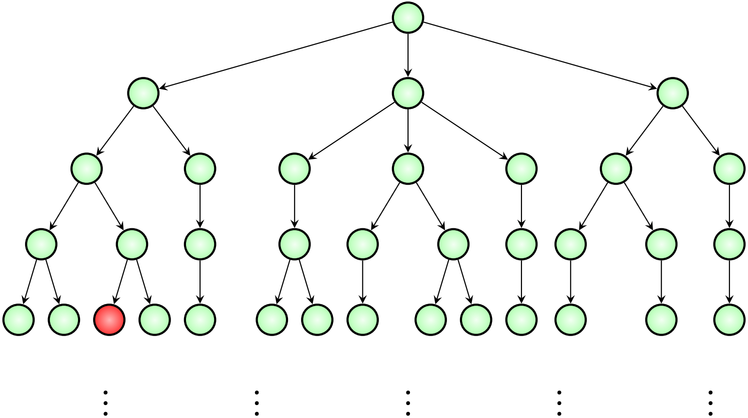

CTL*

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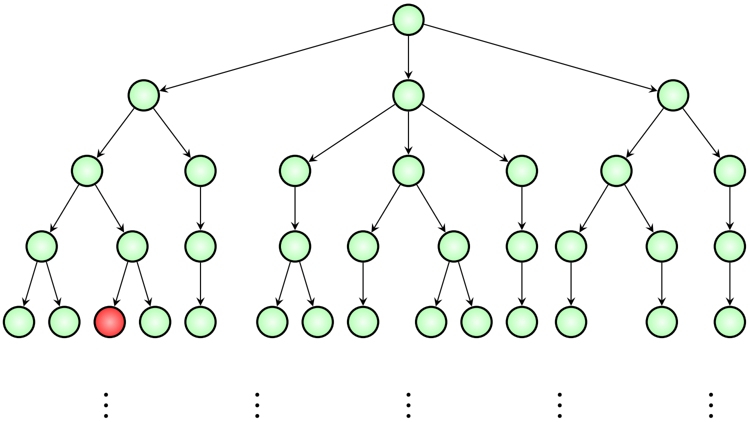
Examples: $\mathrm{E\,F}\,p_1$, $\mathrm{A\,F\,A\,G}\,p_1$, $\mathrm{A\,F\,G}\,p_2$, $\mathrm{A}\,p_1$, $\mathrm{A\,E}\,p_1$

When does a **state** in a tree **satisfy** a state formula?



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$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

- **Every state** satisfies *true*

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$p_i \in AP$ $\quad$ ϕ_1, ϕ_2 : State formulae $\quad$ α : Path formula

- **Every state** satisfies *true*
- State satisfies p_i if its **label contains** p_i

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- State satisfies p_i if its **label contains** p_i
- State satisfies $\phi_1 \wedge \phi_2$ if it satisfies **both** ϕ_1 **and** ϕ_2

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- State satisfies $\neg\,\phi$ if it **does not satisfy** ϕ

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$p_i \in AP$ $\quad \phi_1, \phi_2$: State formulae $\quad \alpha$: Path formula

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- State satisfies $\mathrm{E}\ \alpha$ if there **exists a path** starting from the state satisfying α

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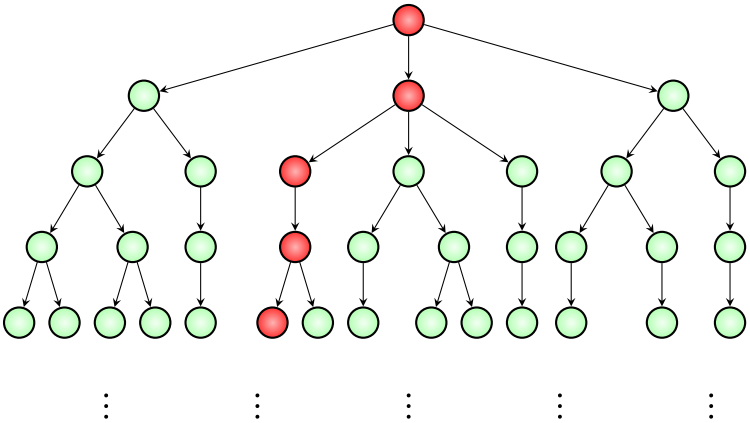
- State satisfies E α if there **exists a path** starting from the state satisfying α
- State satisfies A α if **all paths** starting from the state satisfy α

When does a **path** in a tree **satisfy** a path formula?



Path formulae

$$\alpha := \phi \mid \alpha_1 \wedge \alpha_2 \mid \neg\alpha_1 \mid X\,\alpha_1 \mid \alpha_1\,U\,\alpha_2 \mid F\,\alpha_1 \mid G\,\alpha_1$$

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ϕ : State formula α_1, α_2 : Path formulae

- **Path** satisfies ϕ if the **initial state** of the path satisfies ϕ

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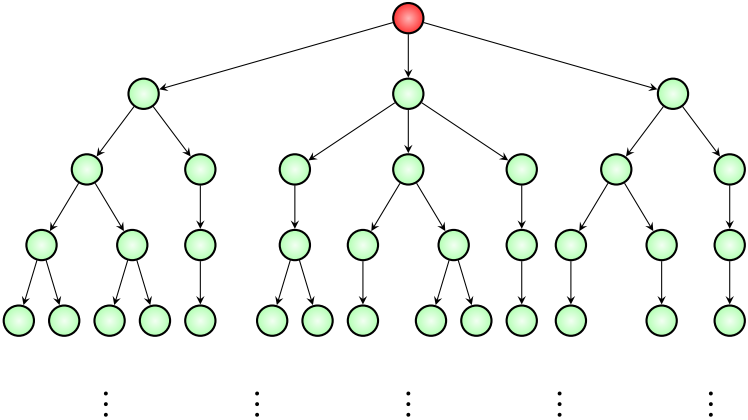
ϕ : State formula α_1, α_2 : Path formulae

- **Path** satisfies ϕ if the **initial state** of the path satisfies ϕ
- Rest **standard** semantics like LTL

A **tree** satisfies **state formula** ϕ if its **root** satisfies ϕ



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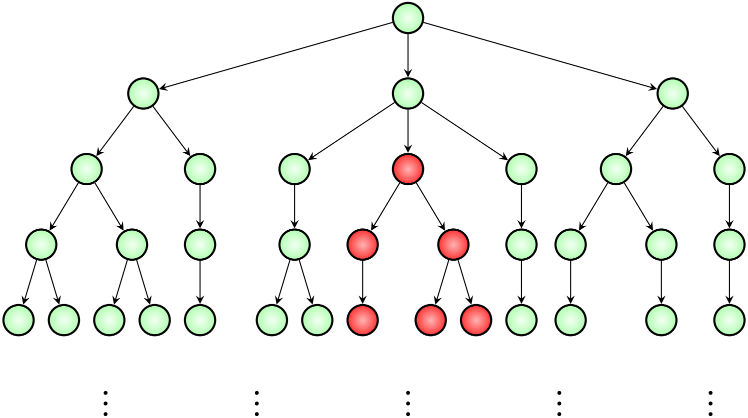
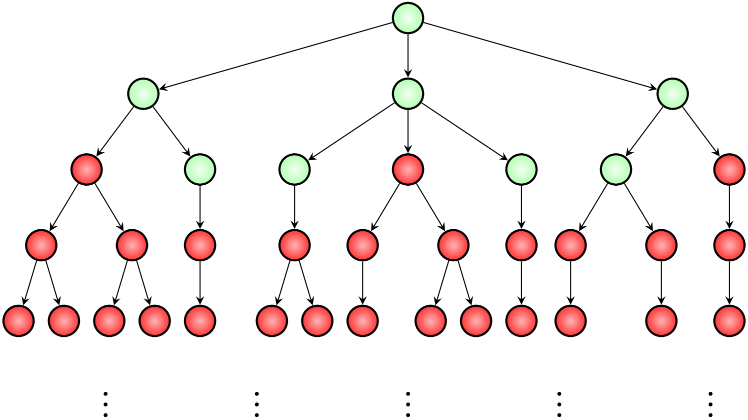
- **A** p_1:
  - All paths satisfy p_1
  - All paths start with p_1
  - Same as p_1!

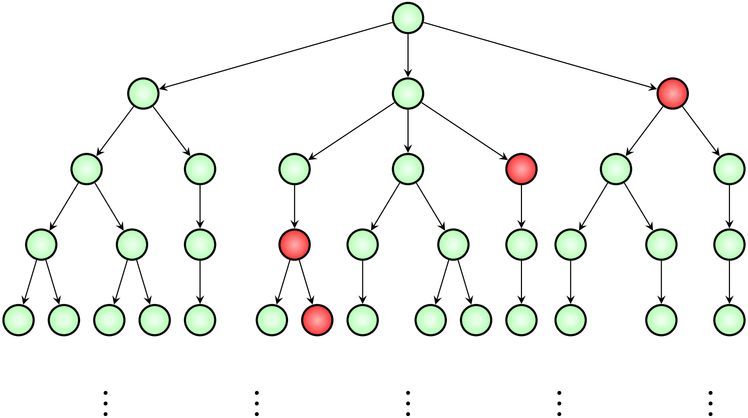
E F A G (*red*)



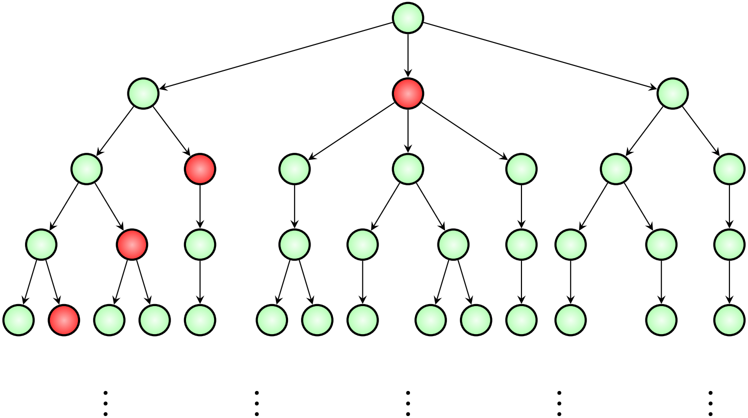
A F A G (*red*)



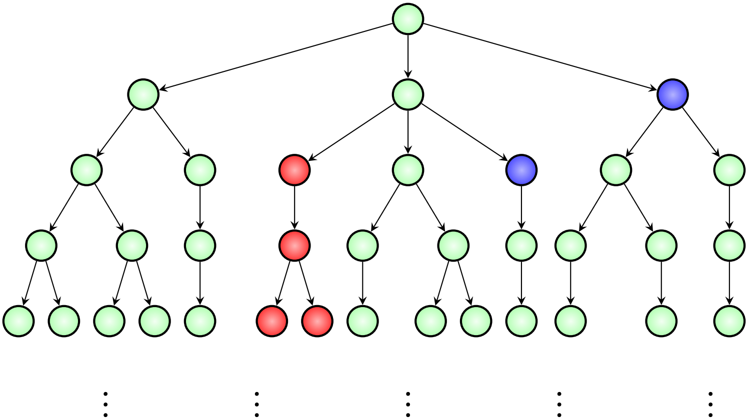
E G E X (*red*)



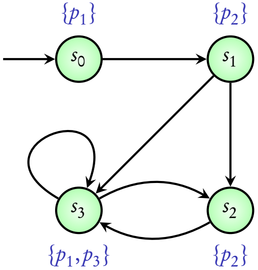
EGEX (*red*)



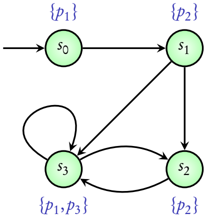
E (E X *blue*) U (A G *red*)

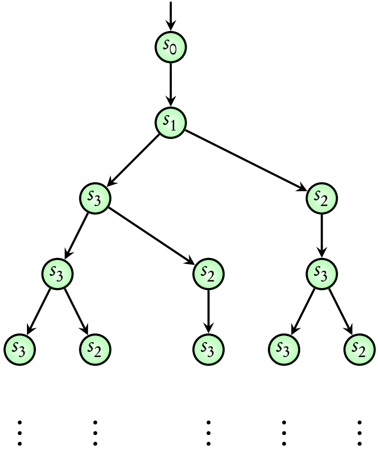


When does a **transition system** satisfy a CTL* formula?



Transition system satisfies CTL* formula ϕ if its **computation tree** satisfies ϕ





Can LTL properties be written using CTL^*?

Transition System (TS) satisfies LTL formula ϕ if

$$\text{Traces(TS)} \subseteq \text{Words}(\phi)$$

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All paths in the **computation tree** of TS satisfy **path formula** ϕ

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Traces(TS) $\subseteq$ Words(ϕ)

All paths in the **computation tree** of TS satisfy **path formula** ϕ

Equivalent CTL* formula: **A** ϕ

Can CTL* properties be written using LTL?

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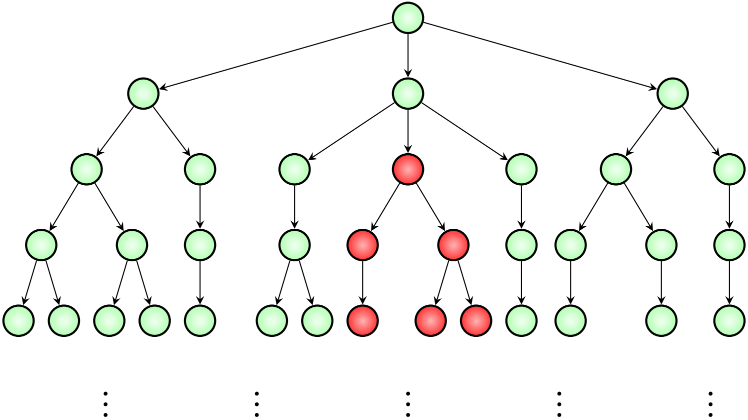
Answer: **No**

E F A G (*red*)



Cannot be expressed in LTL

Summary

CTL*

Syntax and semantics

State formulae, Path formulae

LTL properties $\subseteq$ CTL* properties