Unit-8: Algorithms for LTL

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NPTEL-course

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Module 3:

Automaton construction

$p_1 \ \mathbf{U} \ p_2$

	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_2\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1,p_2\}$	$\cdots$
p_1	1	1	1	1	0	1	1	1	1	
p_2	0	0	0	0	1	0	0	0	1	
$p_1\ U\ p_2$	1	1	1	1	1	1	1	1	1	

	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\cdots$
p_1	1	0	1	0	1	0	1	0	1	
p_2	0	0	0	0	0	0	0	0	0	
$p_1\ U\ p_2$	0	0	0	0	0	0	0	0	0	

$\mathit{true}\ \mathbf{U}\ (\neg p_1 \ \wedge \ \mathbf{X}\ (\neg\ p_2\ \mathbf{U}\ p_1))$

	$\{\}$	$\{p_2\}$	$\{\}$	$\{\}$	$\{p_1\}$	$\{p_1,p_2\}$	$\{p_1,p_2\}$	$\cdots$
p_1	0	0	0	0	1	1	1	
p_2	0	1	0	0	0	1	1	
$\neg p_1$	1	1	1	1	0	0	0	
$\neg p_2$	1	0	1	1	1	0	0	
$\neg p_2\ \mathrm{U}\ p_1$	0	0	1	1	1	1	1	
$\mathrm{X}\ (\neg p_2\ \mathrm{U}\ p_1)$	0	1	1	1	1	1	0	
$\neg p_1\ \wedge\ \mathrm{X}\ (\neg p_2\ \mathrm{U}\ p_1)$	0	1	1	1	0	0	0	
$\mathit{true}\ \mathrm{U}\ (\neg p_1\ \wedge\ \mathrm{X}\ (\neg p_2\ \mathrm{U}\ p_1))$	1	1	1	1	1	1	0	

$\neg\ \mathit{true}\ \mathbf{U}\ \neg(\mathit{true}\ \mathbf{U}\ p_1)$

	$\{\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{\}$
p_1	0	0	1	0	0	1	0	0
true	1	1	1	1	1	1	1	1
$\mathit{true}\ \mathrm{U}\ p_1$	1	1	1	1	1	1	1	1
$\neg\ \mathit{true}\ \mathrm{U}\ p_1$	0	0	0	0	0	0	0	0
$\mathit{true}\ \mathrm{U}\ \neg\ (\mathit{true}\ \mathrm{U}\ p_1)$	0	0	0	0	0	0	0	0
$\neg\ \mathit{true}\ \mathrm{U}\ \neg\ (\mathit{true}\ \mathrm{U}\ p_1)$	1	1	1	1	1	1	1	1

	$\{p_1\}$	$\{p_1\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
p_1	1	1	0	0	0	0	0	0	0
true	1	1	1	1	1	1	1	1	1
$\mathit{true}\ \mathrm{U}\ p_1$	1	1	0	0	0	0	0	0	0
$\neg\ \mathit{true}\ \mathrm{U}\ p_1$	0	0	1	1	1	1	1	1	1
$\mathit{true}\ \mathrm{U}\ \neg\ (\mathit{true}\ \mathrm{U}\ p_1)$	1	1	1	1	1	1	1	1	1
$\neg\ \mathit{true}\ \mathrm{U}\ \neg\ (\mathit{true}\ \mathrm{U}\ p_1)$	0	0	0	0	0	0	0	0	0

p_1 **U** p_2

	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_2\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1\}$	$\{p_1,p_2\}$	$\cdots$
p_1	1	1	1	1	0	1	1	1	1	
p_2	0	0	0	0	1	0	0	0	1	
$p_1\ U\ p_2$	1	1	1	1	1	1	1	1	1	

	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{p_1\}$	$\cdots$
p_1	1	0	1	0	1	0	1	0	1	
p_2	0	0	0	0	0	0	0	0	0	
$p_1\ U\ p_2$	0	0	0	0	0	0	0	0	0	

true **U** $(\neg p_1 \wedge$ **X** $(\neg p_2$ **U** $p_1))$

	$\{\}$	$\{p_2\}$	$\{\}$	$\{\}$	$\{p_1\}$	$\{p_1,p_2\}$	$\{p_1,p_2\}$	$\cdots$
p_1	0	0	0	0	1	1	1	
p_2	0	1	0	0	0	1	1	
$\neg p_1$	1	1	1	1	0	0	0	
$\neg p_2$	1	0	1	1	1	0	0	
$\neg p_2$ U p_1	0	0	1	1	1	1	1	
X $(\neg p_2$ U $p_1)$	0	1	1	1	1	1	0	
$\neg p_1 \wedge$ X $(\neg p_2$ U $p_1)$	0	1	1	1	0	0	0	
true U $(\neg p_1 \wedge$ X $(\neg p_2$ U $p_1))$	1	1	1	1	1	1	0	

$\neg$ *true* **U** $\neg($*true* **U** $p_1)$

	$\{\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{\}$	$\{p_1\}$	$\{\}$	$\{\}$
p_1	0	0	1	0	0	1	0	0
true	1	1	1	1	1	1	1	1
true U p_1	1	1	1	1	1	1	1	1
$\neg$ *true* U p_1	0	0	0	0	0	0	0	0
true U $\neg$ (*true* U p_1)	0	0	0	0	0	0	0	0
$\neg$ *true* U $\neg$ (*true* U p_1)	1	1	1	1	1	1	1	1

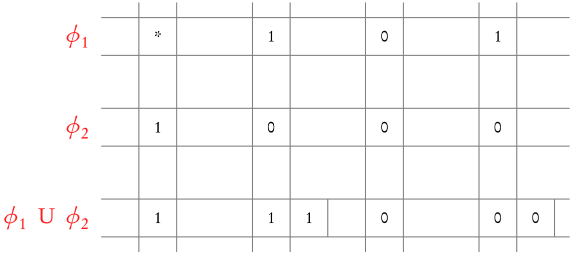
	$\{p_1\}$	$\{p_1\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
p_1	1	1	0	0	0	0	0	0	0
true	1	1	1	1	1	1	1	1	1
true U p_1	1	1	0	0	0	0	0	0	0
$\neg$ *true* U p_1	0	0	1	1	1	1	1	1	1
true U $\neg$ (*true* U p_1)	1	1	1	1	1	1	1	1	1
$\neg$ *true* U $\neg$ (*true* U p_1)	0	0	0	0	0	0	0	0	0

Construct an automaton with **states as column vectors** that can **guess accepting expansions**

Example 1: $p_1 \mathrm{U} p_2$

p_1	0	0	0	0	1	1	1	1
p_2	0	0	1	1	0	0	1	1
$p_1 \, \mathrm{U} \, p_2$	0	1	0	1	0	1	0	1

p_1	0	0	0	0	1	1	1	1
p_2	0	0	1	1	0	0	1	1
$p_1 \, \mathrm{U} \, p_2$	0	1	0	1	0	1	0	1



Recall Until-compatibility

p_1	0	0	0	1	1	1	1
p_2	0	1	1	0	0	1	1
$p_1 \mathrm{U} p_2$	0	0	1	0	1	0	1

ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
$\phi_1 \ \mathrm{U} \ \phi_2$	1	1	1	0	0	0

Recall Until-compatibility

p_1	0	0	1	1	1	1
p_2	0	1	0	0	1	1
$p_1 \mathrm{U}\, p_2$	0	1	0	1	0	1

ϕ_1	*		1			0		1	
ϕ_2	1		0			0		0	
$\phi_1 \ \mathrm{U} \ \phi_2$	1		1	1		0		0	0

Recall Until-compatibility

p_1	0	0	1	1	1
p_2	0	1	0	0	1
p_1 U p_2	0	1	0	1	1

ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

Recall Until-compatibility

p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \mathbin{\mathrm{U}} p_2$	0	1	0	1	1

Compatible states

ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
$\phi_1 \mathbin{\mathrm{U}} \phi_2$	1	1	1	0	0	0

Recall Until-compatibility

p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \mathrm{U}\, p_2$	0	1	0	1	1

q_0

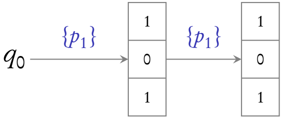
p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \mathrm{U} p_2$	0	1	0	1	1

$$q_0 \xrightarrow{\{p_1\}}$$

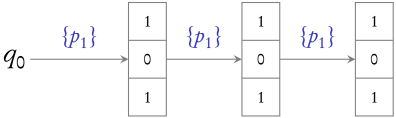
p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \mathbin{\mathrm{U}} p_2$	0	1	0	1	1

$$q_0 \xrightarrow{\{p_1\}} \begin{bmatrix}1\\0\\1\end{bmatrix}$$

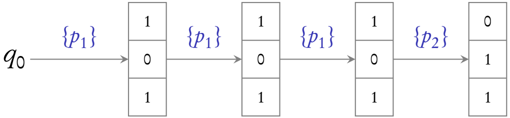
p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \mathrm{U}\, p_2$	0	1	0	1	1



p_1	0	0	1	1	1
p_2	0	1	0	0	1
p_1 U p_2	0	1	0	1	1

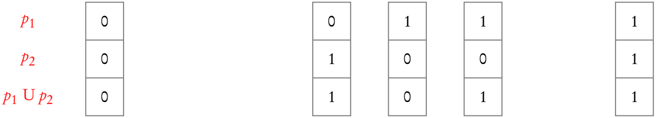


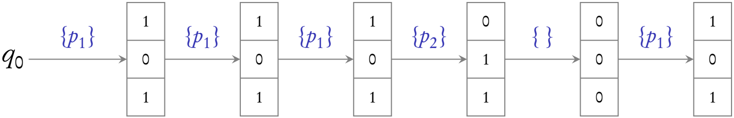
p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \cup p_2$	0	1	0	1	1

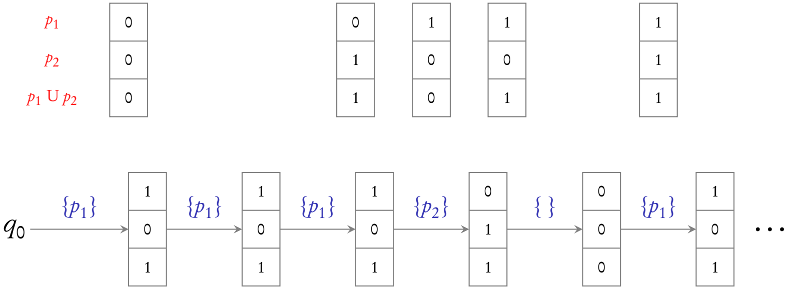


p_1	0			0	1	1		1
p_2	0			1	0	0		1
$p_1 \cup p_2$	0			1	0	1		1

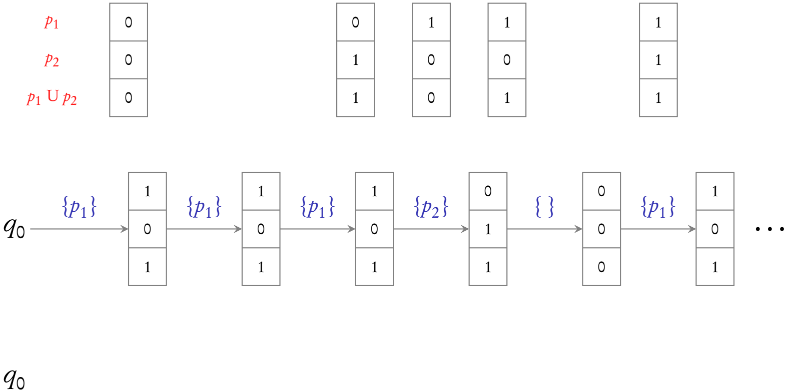
$$q_0 \xrightarrow{\{p_1\}} \begin{bmatrix}1\\0\\1\end{bmatrix} \xrightarrow{\{p_1\}} \begin{bmatrix}1\\0\\1\end{bmatrix} \xrightarrow{\{p_1\}} \begin{bmatrix}1\\0\\1\end{bmatrix} \xrightarrow{\{p_2\}} \begin{bmatrix}0\\1\\1\end{bmatrix} \xrightarrow{\{\,\}} \begin{bmatrix}0\\0\\0\end{bmatrix}$$

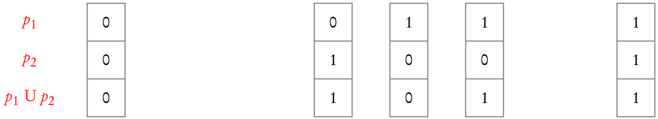

p_1
p_2
p_1 U p_2
0 0 0
0 1 1
1 0 0
1 0 1
1 1 1

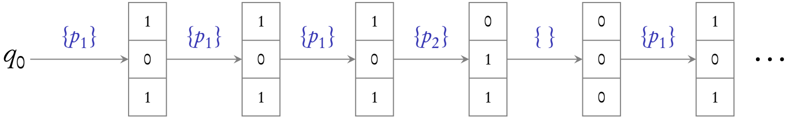

q_0
$\{p_1\}$
1 0 1
$\{p_1\}$
1 0 1
$\{p_1\}$
1 0 1
$\{p_2\}$
0 1 1
$\{\}$
0 0 0
$\{p_1\}$
1 0 1

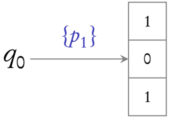


p_1
p_2
$p_1 \cup p_2$
0 0 0
0 1 1
1 0 0
1 0 1
1 1 1
q_0
$\{p_1\}$
1 0 1
$\{p_1\}$
1 0 1
$\{p_1\}$
1 0 1
$\{p_2\}$
0 1 1
$\{\,\}$
0 0 0
$\{p_1\}$
1 0 1
...

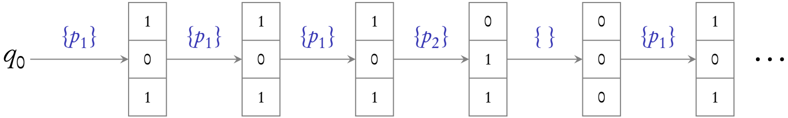

p_1
p_2
$p_1 \cup p_2$
q_0
$\{p_1\}$
$\{p_1\}$
$\{p_1\}$
$\{p_2\}$
$\{\,\}$
$\{p_1\}$
$\cdots$
q_0

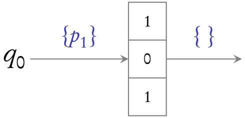

p_1
p_2
$p_1 \cup p_2$
0 0 0
0 1 1
1 0 0
1 0 1
1 1 1


q_0
$\{p_1\}$
$\{p_1\}$
$\{p_1\}$
$\{p_2\}$
$\{\,\}$
$\{p_1\}$
1 0 1
1 0 1
1 0 1
0 1 1
0 0 0
1 0 1
$\cdots$

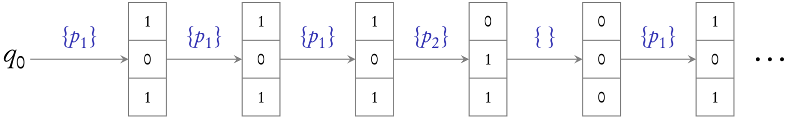

q_0
$\{p_1\}$
1 0 1

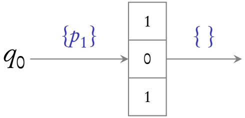
p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \cup p_2$	0	1	0	1	1



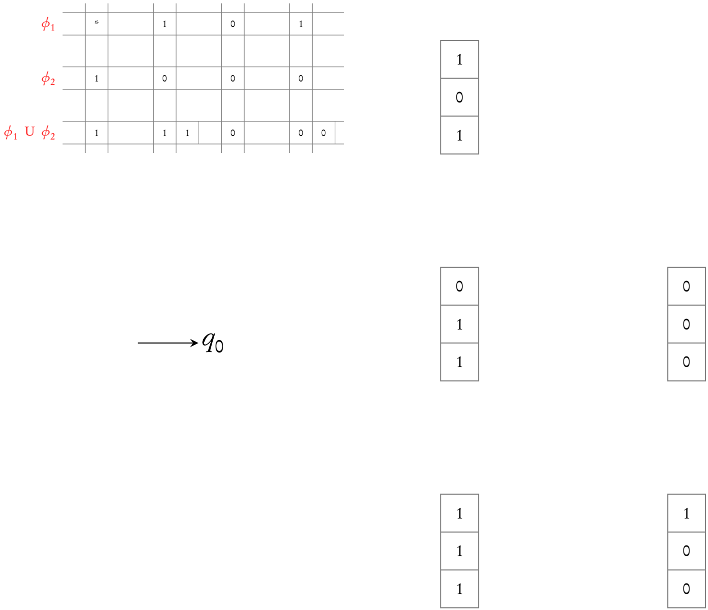


p_1	0	0	1	1	1
p_2	0	1	0	0	1
$p_1 \cup p_2$	0	1	0	1	1

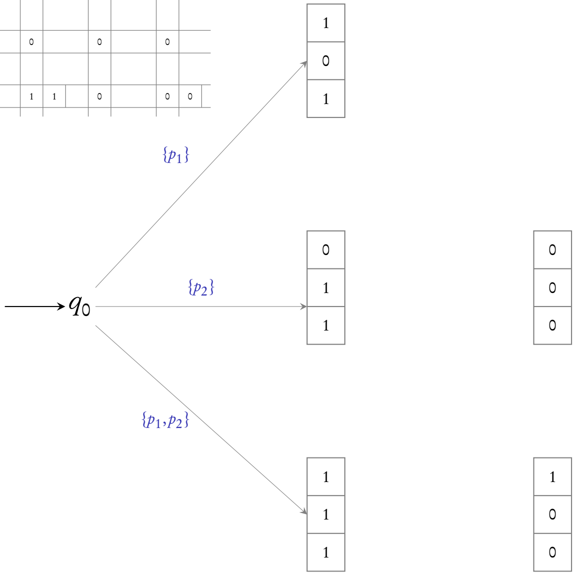




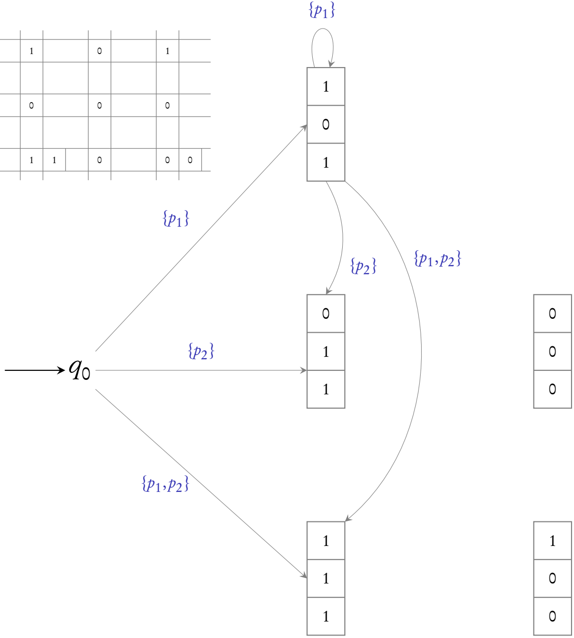
No compatible transition


ϕ_1
*
1
0
1
ϕ_2
1
0
0
0
ϕ_1 U ϕ_2
1
1
1
0
0
0
q_0
1
0
1
0
1
1
0
0
0
1
1
1
1
0
0

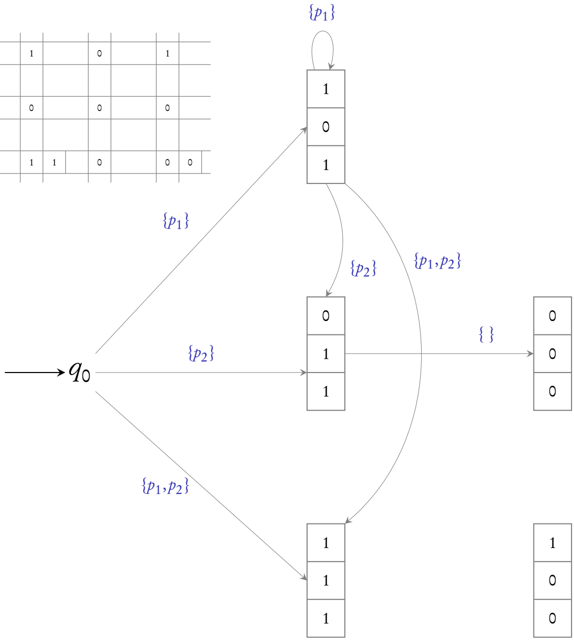
ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

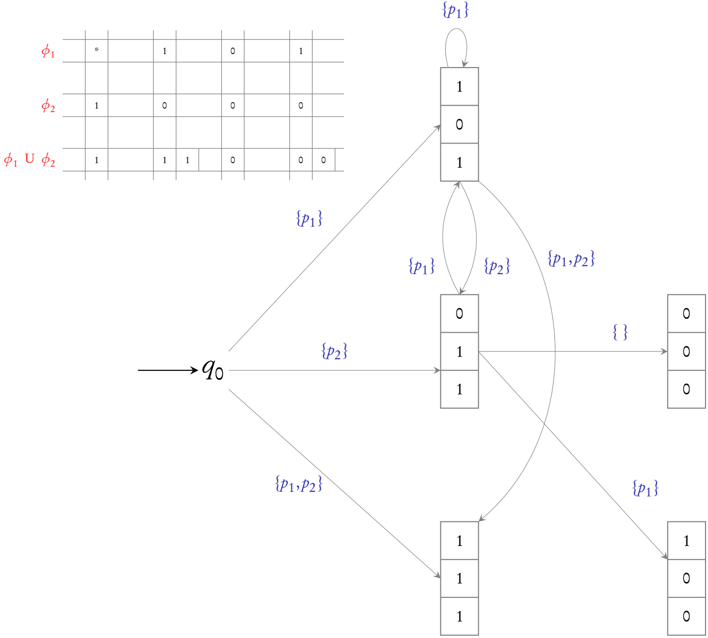


ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

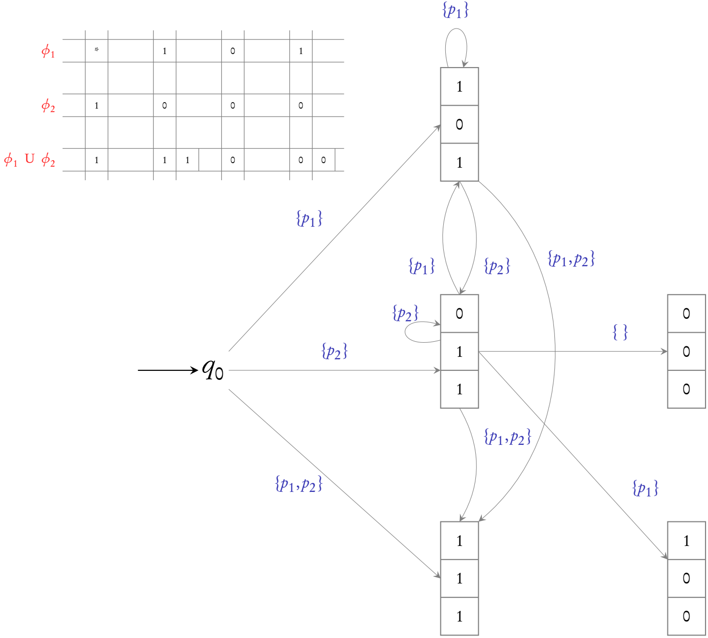


ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

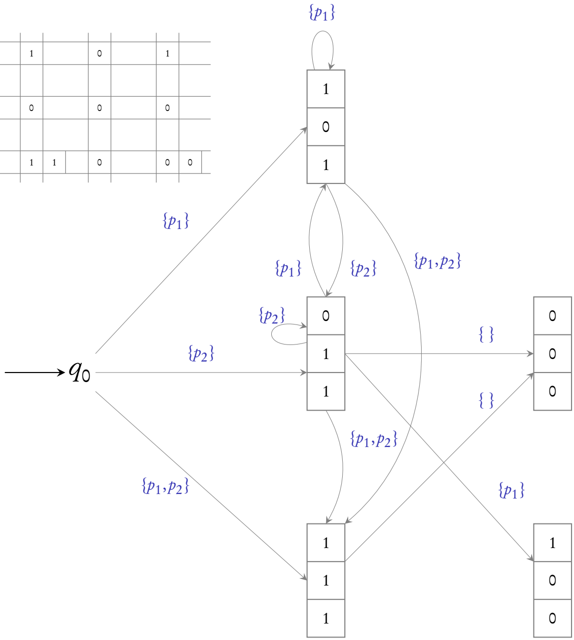




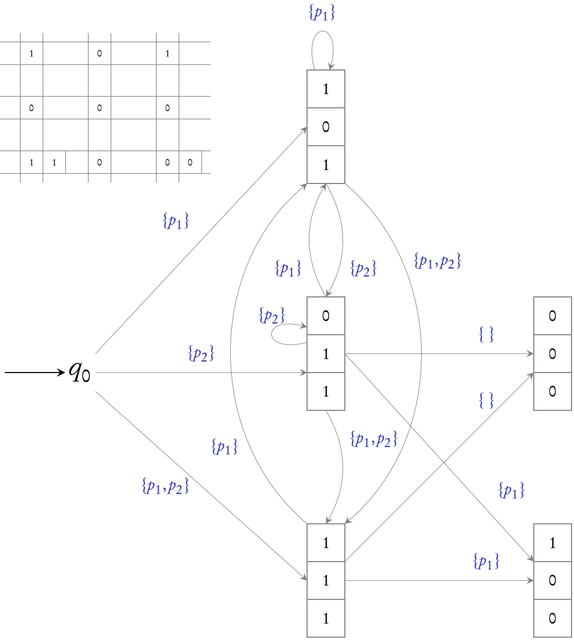
ϕ_1
ϕ_2
ϕ_1 U ϕ_2
*
1
0
q_0
$\{p_1\}$
$\{p_2\}$
$\{p_1, p_2\}$
$\{\}$

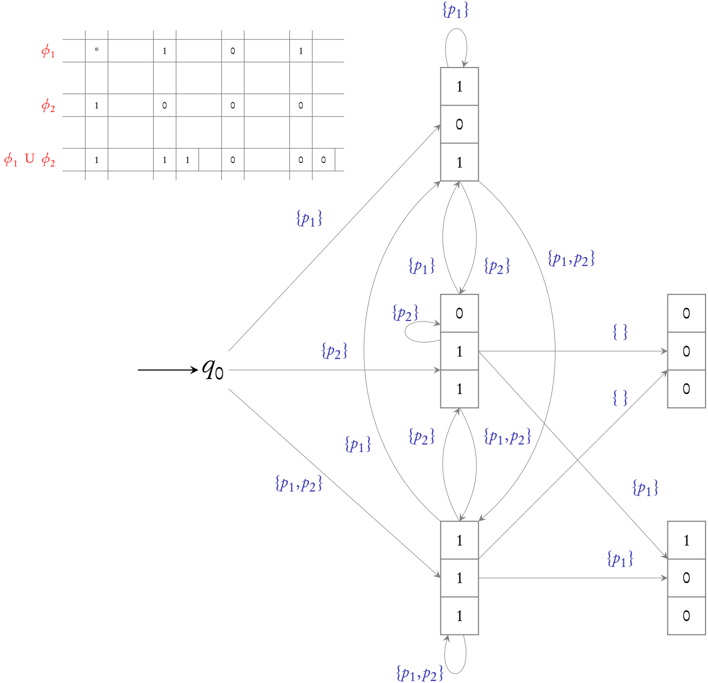

ϕ_1
ϕ_2
ϕ_1 U ϕ_2
*
1
0
q_0
$\{p_1\}$
$\{p_2\}$
$\{p_1, p_2\}$
$\{\,\}$

ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

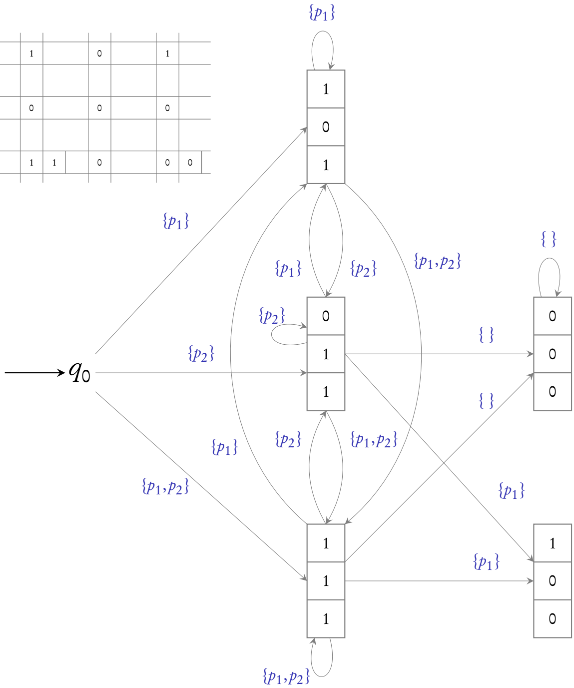


ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0

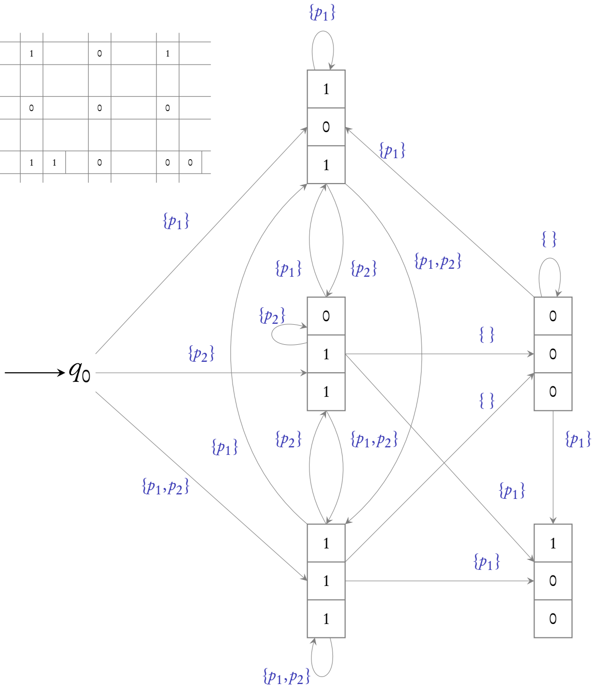



ϕ_1
ϕ_2
ϕ_1 U ϕ_2
* 1 0 1
1 0 0 0
1 1 1 0 0 0
q_0
$\{p_1\}$
$\{p_2\}$
$\{p_1,p_2\}$
$\{\}$

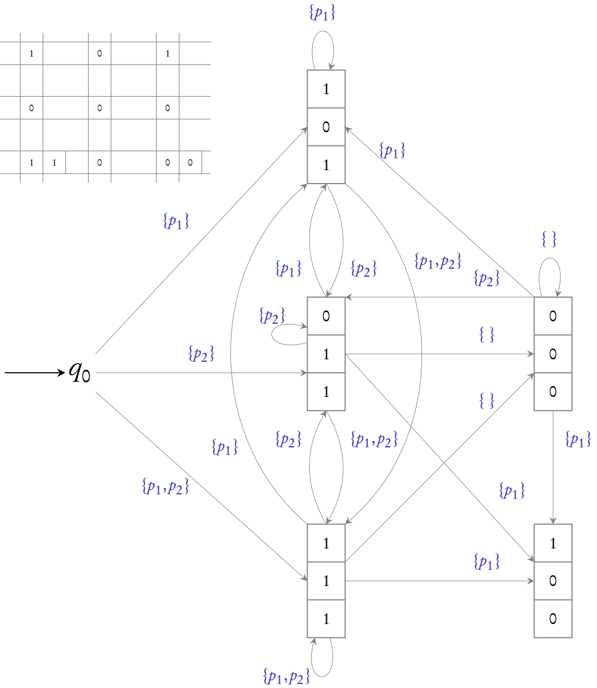
ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



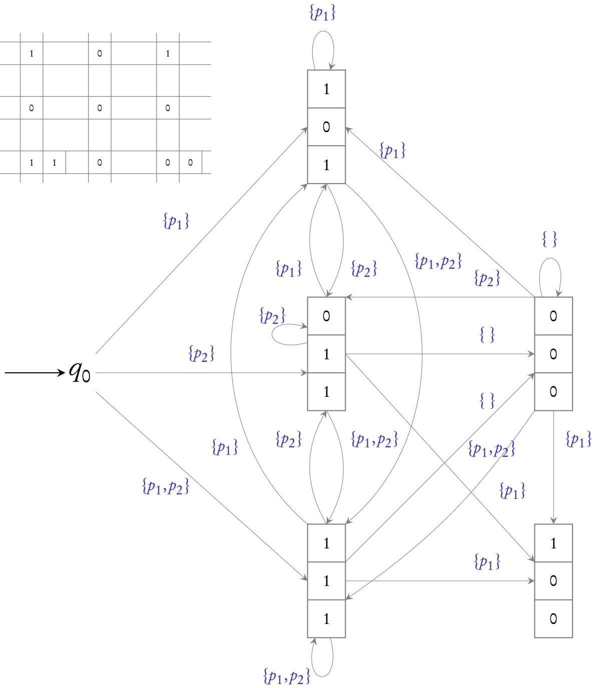
ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



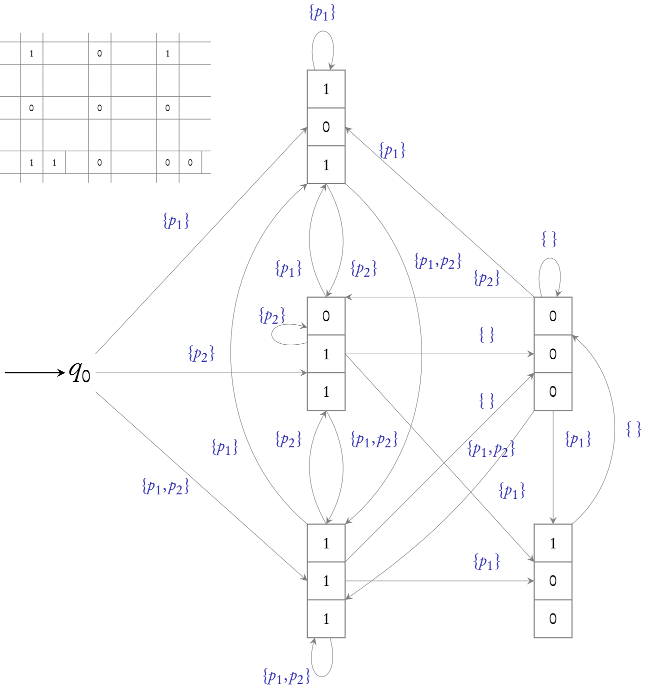
ϕ_1	*	1	0	1
ϕ_2	1	0	0	0
ϕ_1 U ϕ_2	1	1 1	0	0 0



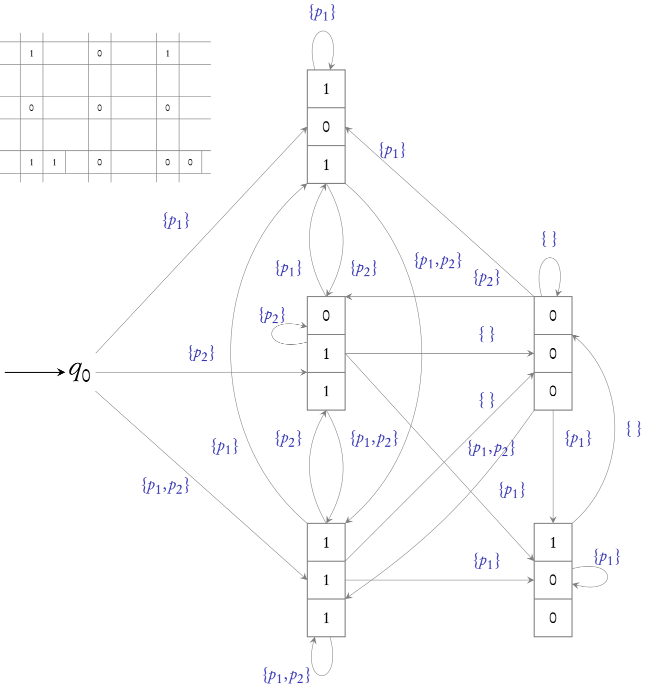
ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



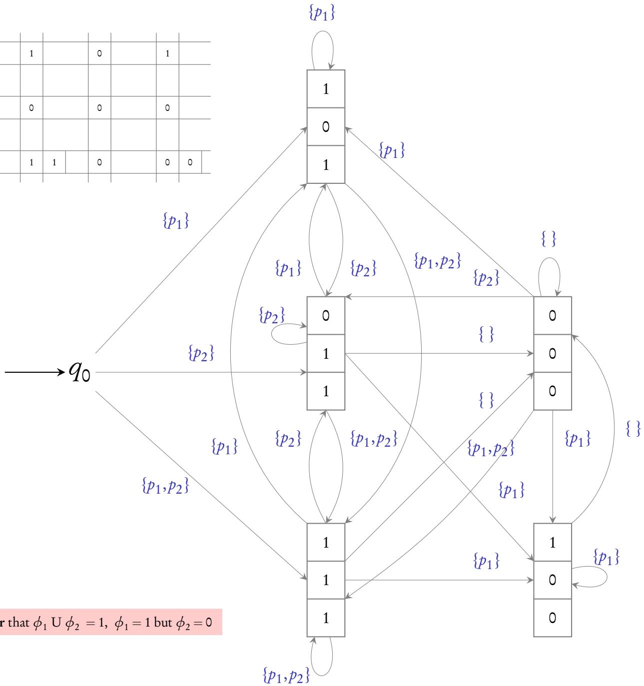
ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



ϕ_1	*	1	0	1
ϕ_2	1	0	0	0
$\phi_1 \; \mathrm{U} \; \phi_2$	1	1 1	0	0 0

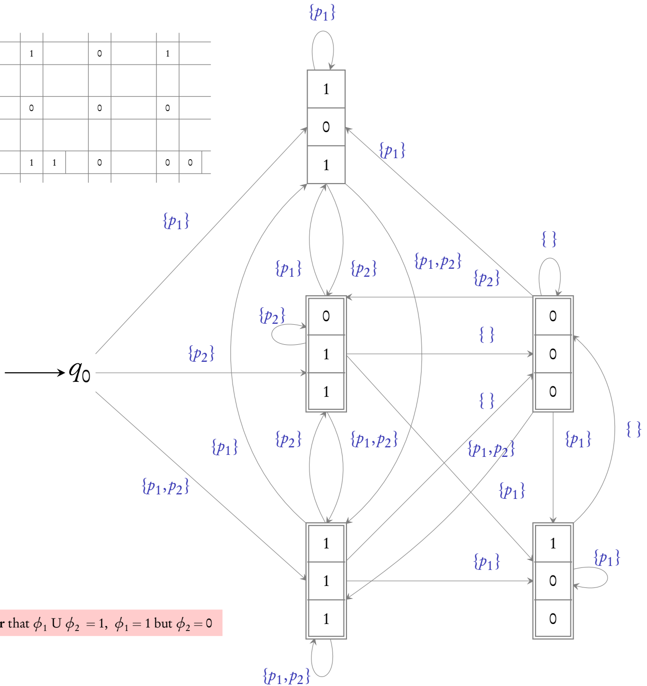


ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



Cannot happen forever that ϕ_1 U ϕ_2 $= 1$, $\phi_1 = 1$ but $\phi_2 = 0$

ϕ_1	*	1		0	1	
ϕ_2	1	0		0	0	
ϕ_1 U ϕ_2	1	1	1	0	0	0



Cannot happen forever that ϕ_1 U $\phi_2 = 1$, $\phi_1 = 1$ but $\phi_2 = 0$

Example 2: $(\mathrm{X}\ p_1)\ \mathrm{U}\ p_2$

p_1	*
p_2	*
$\mathrm{X}\,p_1$	*
$(\mathrm{X}\,p_1)\,\mathrm{U}\,p_2$	*

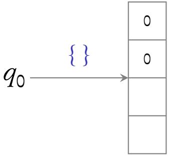
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\, \mathrm{U}\, p_2$	*

q_0

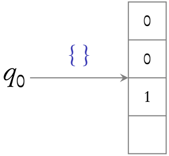
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\ \mathrm{U}\ p_2$	*

$$q_0 \xrightarrow{\{\ \}}$$

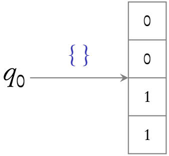
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\ \mathrm{U}\ p_2$	*



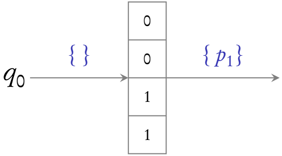
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\ \mathrm{U}\ p_2$	*



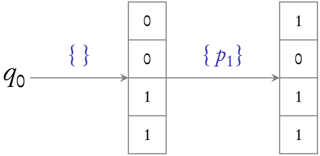
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\, \mathrm{U}\, p_2$	*



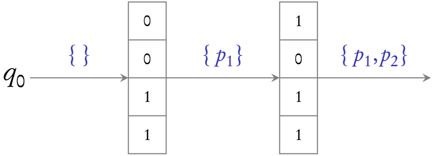
p_1	*
p_2	*
$X\,p_1$	*
$(X\,p_1)\;U\;p_2$	*



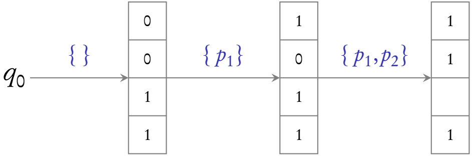
p_1	*
p_2	*
$X\,p_1$	*
$(X\,p_1)\,U\,p_2$	*



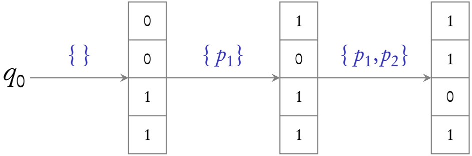
p_1	*
p_2	*
$X\, p_1$	*
$(X\, p_1)\ U\ p_2$	*



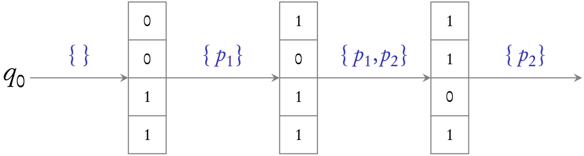
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\ \mathrm{U}\ p_2$	*



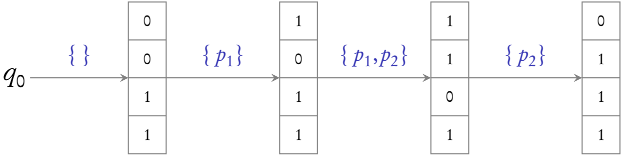
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\, \mathrm{U}\, p_2$	*



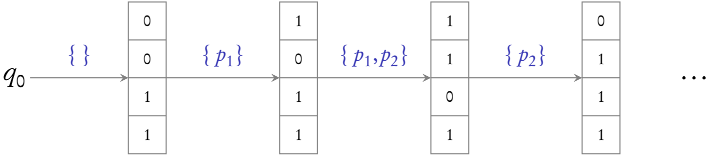
p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\, \mathrm{U}\, p_2$	*



p_1	*
p_2	*
$X\, p_1$	*
$(X\, p_1)\ U\ p_2$	*



p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1)\ \mathrm{U}\ p_2$	*



Coming next: Construction for an arbitrary LTL formula ϕ

Step 1: List down subformulae of ϕ

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p_1	*
p_2	*
$p_1 \mathbin{\mathrm{U}} p_2$	*

p_1	*
p_2	*
$\mathrm{X}\, p_1$	*
$(\mathrm{X}\, p_1) \mathbin{\mathrm{U}} p_2$	*

Step 2: Check **AND-NOT** and **Until** compatibility

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p_1	0
p_2	0
$p_1 \mathbin{\mathrm{U}} p_2$	1

p_1	0
p_2	1
$\mathrm{X}\, p_1$	0
$(\mathrm{X}\, p_1) \mathbin{\mathrm{U}} p_2$	0

Incompatible states!

Step 2: Check **AND-NOT** and **Until** compatibility

p_1	0
p_2	0
$p_1 \mathbin{\mathrm{U}} p_2$	1

p_1	0
p_2	1
$\mathrm{X}\, p_1$	0
$(\mathrm{X}\, p_1) \mathbin{\mathrm{U}} p_2$	0

Incompatible states!

Remove incompatible states and **add** a new state $\{q_0\}$

Step 3: Add transitions satisfying

Word, **X** and **Until** compatibility

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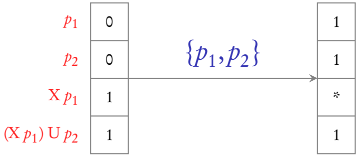

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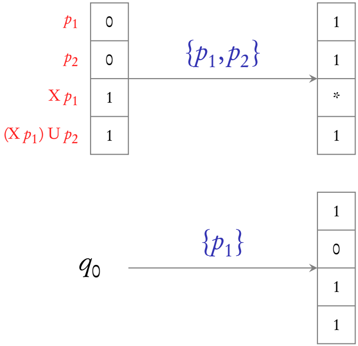
Word, **X** and **Until** compatibility



From q_0 add compatible transitions to states where last entry is 1

Step 4: Accepting states should ensure Until-eventuality condition

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For **every** Until subformula $\phi_1 \mathrm{U} \phi_2$, define

$F_{\phi_1 \mathrm{U} \phi_2}$: set of states where $\phi_1 \mathrm{U} \phi_2 = 0$ or $\phi_2 = 1$

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$F_{p_1 \, \mathrm{U} \, p_2}$:

0	1	0	1
0	0	1	1
0	0	1	1

Final automaton $\mathscr{A}_\phi$

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- Compatible **states** $+ q_0$

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- Compatible **transitions**

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In general, this algorithm gives NBA which is **exponential** in size of formula