Unit-7: Linear Temporal Logic

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NPTEL-course

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Module 2:

Semantics of LTL

AP-INF = set of **infinite words** over *PowerSet*(**AP**)

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Property 1: p_1 is always true

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$$\{ A_0 A_1 A_2 \cdots \in \text{AP-INF} \mid \textbf{ each } A_i \text{ contains } p_1 \}$$

$\{p_1\}\{p_1\}\{p_1\}\{p_1\}\{p_1\}\{p_1\}\{p_1\}\ldots$

$\{p_1\}\{p_1,p_2\}\{p_1\}\{p_1,p_2\}\{p_1\}\{p_1,p_2\}\ldots$

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$$\{ A_0 A_1 A_2 \cdots \in \text{AP-INF} \mid \textbf{exists } A_i \text{ containing } p_1 \text{ and } \textbf{every } A_j \text{ contains } p_2 \}$$

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LTL can be used to **describe subsets** of AP-INF

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid X\,\phi \mid \phi_1\, U\, \phi_2$$

LTL formula ϕ $\longrightarrow$ Words(ϕ)

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LTL formula ϕ $\longrightarrow$ Words(ϕ) $\subseteq$ AP-INF

Words(ϕ): set of words in AP-INF that **satisfy** ϕ

When does a **word satisfy LTL formula** ϕ?

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid X\,\phi \mid \phi_1\,U\,\phi_2$$

Word $\sigma : A_0 A_1 A_2 \ \ldots \ \in \text{AP-INF}$

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for all $0 \leq i < j$ $A_i A_{i+1} \ldots$ **satisfies** ϕ_1

$$\textbf{Words}(\phi) \;=\; \{\, \sigma \in \text{AP-}\textsc{inf} \mid \sigma \textbf{ satisfies } \phi \,\}$$

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and for all $0 \leq i < j$ $A_i A_{i+1} \ldots$ **satisfies** $true$

F ϕ: *true* U ϕ

σ **satisfies** *true* U ϕ if there exists j s.t. $A_j A_{j+1} \ldots$ **satisfies** ϕ

F ϕ: $\quad$ $true\ U\ \phi$

σ **satisfies** $true\ U\ \phi$ $\quad$ if $\quad$ there exists j s.t. $A_j A_{j+1} \dots$ **satisfies** ϕ

G ϕ: $\quad$ $\neg\, F \neg\, \phi$

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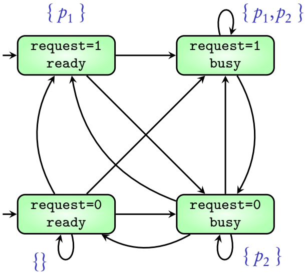
$\mathrm{AP} = \{\ p_1,\ p_2\ \}$

Transition System

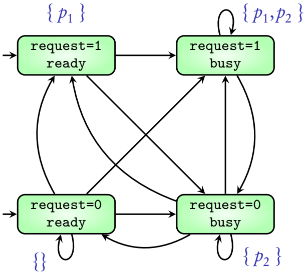


Property

LTL formula ϕ

$$\mathrm{AP} = \{\ p_1,\ p_2\ \ \}$$

Transition System

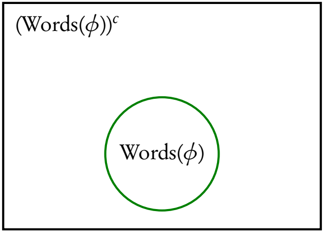


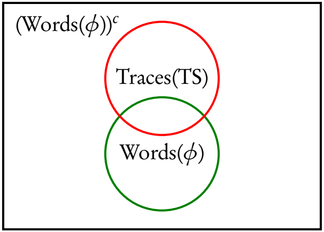
Property

LTL formula ϕ

Transition system TS satisfies formula ϕ if

Traces(TS) $\subseteq$ Words(ϕ)


(Words(ϕ))^c
Words(ϕ)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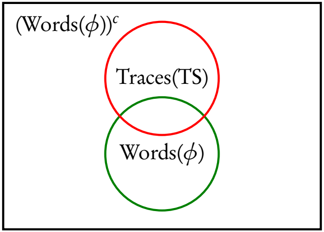
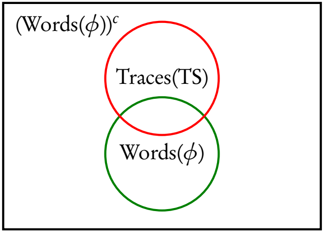


TS **does not satisfy** ϕ TS **does not satisfy** $\neg\phi$



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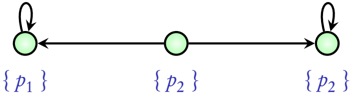

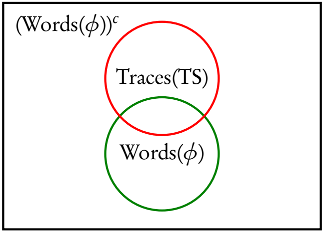

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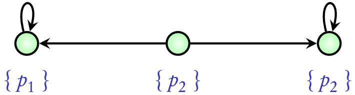

Above TS **does not satisfy** $F\ p_1$ Above TS **does not satisfy** $\neg F\ p_1$

Semantics of LTL