Unit-5: ω-regular properties

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Module 2:

ω-regular expressions

Languages over finite words

Σ : finite alphabet $\Sigma^* =$ set of **all words** over Σ

Language: A set of **finite words**

$\{ ab, abab, ababab, \ldots \}$

finite words starting with an a

finite words starting with a b

$\{ \epsilon, b, bb, bbb, \ldots \}$

$\{ \epsilon, ab, abab, ababab, \ldots \}$

$\{ \epsilon, bbb, bbbbbb, (bbb)^3, \ldots \}$

words starting and ending with an a

$\{ \epsilon, ab, aabb, aaabbb, a^4 b^4 \ldots \}$

Regular expressions

Σ : finite alphabet $\quad \Sigma^* \; = \;$ set of **all words** over Σ

Language: A set of **finite words**

$ab(ab)^*$ $\quad \{\, ab, abab, ababab, \ldots \}$

$a\Sigma^*$ $\quad$ finite words starting with an a

$b\Sigma^*$ $\quad$ finite words starting with a b

b^* $\quad \{\, \epsilon, b, bb, bbb, \ldots \}$

$(ab)^*$ $\quad \{\, \epsilon, ab, abab, ababab, \ldots \}$

$(bbb)^*$ $\quad \{\, \epsilon, bbb, bbbbbb, (bbb)^3, \ldots \}$

$a\Sigma^*a$ $\quad$ words starting and ending with an a

$\{\, \epsilon, ab, aabb, aaabbb, a^4b^4 \ldots \}$

Alphabet $\Sigma = \{ a, b \}$

$$\Sigma \cdot \Sigma = \{ a, b \} \cdot \{ a, b \} = \{ aa, ab, ba, bb \}$$

$$aba \cdot \epsilon = aba$$

$$\epsilon \cdot bbb = bbb$$

$$w \cdot \epsilon = w$$

$$\epsilon \cdot w = w$$

$\Sigma^0 = \{ \epsilon \}$ (empty word, with length 0)

$\Sigma^1 =$ words of length 1

$\Sigma^2 =$ words of length 2

$\Sigma^3 =$ words of length 3

$\vdots$

$\Sigma^k =$ words of length k

$\vdots$

$\Sigma^* = \bigcup_{i \geq 0} \Sigma^i$

$=$ **set of all finite length words**

Regular expressions

Regular expressions

ϵ

Regular expressions

$$\epsilon \mid a \mid b$$

Regular expressions

$$\epsilon \;\mid\; a \;\mid\; b \;\mid r_1\, r_2$$

Regular expressions

$$\epsilon \mid a \mid b \mid r_1\, r_2 \mid r_1 + r_2$$

Regular expressions

$$\epsilon \mid a \mid b \mid r_1\, r_2 \mid r_1 + r_2 \mid r^*$$

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where r_1, r_2, r are regular expressions themselves

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$$a^* + b^*$$
$$ab + bb + baa$$
$$(a+b)^* ab(ba+bb)$$
$$(ab+bb)^*$$
$$\vdots$$

Theorem

1. Every **regular expression can be converted to an NFA** accepting the language of the expression
2. Every **NFA can be converted to a regular expression** describing the language of the NFA

Coming next: Languages over **infinite words**

$$\Sigma = \{ a, b \}$$

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Example 1: Infinite word consisting only of *a*

$$\{ aaaaaaaaaaaaaaaa \ldots \}$$

$$\Sigma = \{\, a,\ b \,\}$$

Example 1: Infinite word consisting only of a

$$\{\, aaaaaaaaaaaaaaaa\ldots \,\}$$

Example 2: Infinite words containing only a or only b

$$\{\, aaaaaaaaaaaaaaa\ldots\ ,\ \ bbbbbbbbbbb\ldots \,\}$$

$$\Sigma = \{ a, b \}$$

Example 1: Infinite word consisting only of a

$$\{ aaaaaaaaaaaaaaa\ldots \}$$

Example 2: Infinite words containing only a or only b

$$\{ aaaaaaaaaaaaaaa\ldots , \; bbbbbbbbbbb\ldots \}$$

Example 3: a word in $aa\Sigma^*aa$ followed by only b-s

$$\{ aaaabbbbbb\ldots , \; aababaabbbbb\ldots , \; aabbbbaabbbbbbb\ldots , \; \ldots \}$$

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Example 4: Infinite words where b occurs **only finitely often**

$$\{ aaaaaaaaaaaaaaaa\ldots, \ baaaaaaaaa\ldots, \ babbaaaaaaaaaaa\ldots, \ \ldots \}$$

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Example 5: Infinite words where b occurs **infinitely often**

$$\{ abababababab \dots , \; bbbabbbabbbabbba \dots , \; bbbbbbbbbbbb \dots , \; \dots \}$$

$$\Sigma = \{ a, b \}$$

Example 1: Infinite word consisting only of a $\qquad a^\omega$

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$$\Sigma = \{ a, b \}$$

Example 1: Infinite word consisting only of a a^ω

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Example 2: Infinite words containing only a or only b $a^\omega + b^\omega$

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Example 3: a word in $aa\Sigma^*aa$ followed by only b-s $\quad aa\Sigma^*aa \cdot b^\omega$

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Example 4: Infinite words where b occurs **only finitely often** $(a+b)^* \cdot b^\omega$

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ω-regular expressions

$$G = E_1 \cdot F_1^{\omega} + E_2 \cdot F_2^{\omega} + \cdots + E_n \cdot F_n^{\omega}$$

$E_1, \ldots, E_n, F_1, \ldots, F_n$ are **regular expressions**

and $\epsilon \notin L(F_i)$ for all $1 \leq i \leq n$

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and $\epsilon \notin L(F_i)$ for all $1 \leq i \leq n$

$$L(F^\omega) = \{ w_1 w_2 w_3 \ldots \mid \textbf{each } w_i \in L(F) \}$$

More examples

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- $(a+b)^*c(a+b)^\omega$ words with a **single occurrence** of c

More examples

- $(a+b)^\omega$ set of **all infinite words**
- $a(a+b)^\omega$ infinite words **starting with** an a
- $(a+bc+c)^\omega$ words where every b is **immediately followed** by c
- $(a+b)^*c(a+b)^\omega$ words with a **single occurrence** of c
- $((a+b)^*c)^\omega$ words where c **occurs infinitely often**

$$\begin{aligned}
\mathbf{AP} &= \{\ p_1, p_2,\ \cdots\ , p_k\ \} \\
\Sigma\ =\ \mathit{PowerSet}(\mathbf{AP}) &= \{\ \{\ \},\ \{p_1\},\ \ldots\ ,\ \{p_k\}, \\
&\quad \{\ p_1, p_2\ \},\ \{\ p_1, p_3\ \},\ \ldots\ ,\ \{\ p_{k-1}, p_k\ \}, \\
&\quad \ldots \\
&\quad \{\ p_1, p_2,\ \ldots,\ p_k\ \}\ \}
\end{aligned}$$

A property is a **language of infinite words** over alphabet Σ

$$
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&\qquad\qquad \ldots \\
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\end{aligned}
$$

A property is a **language of infinite words** over alphabet Σ

The property is ω-regular if it can be **described by an ω-regular expression**

$$\mathbf{AP} = \{ \text{wait}, \text{crit} \}$$

$$\Sigma = \mathit{PowerSet}(\mathbf{AP}) = \{ \{ \}, \{\text{wait}\}, \{\text{crit}\}, \{\text{wait}, \text{crit}\} \}$$

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$$\Sigma = \mathit{PowerSet}(\mathbf{AP}) = \{ \{ \}, \{\text{wait}\}, \{\text{crit}\}, \{\text{wait}, \text{crit}\} \}$$

Property: Process enters critical section infinitely often

$$\mathbf{AP} = \{ \text{wait, crit} \}$$

$$\Sigma = \mathit{PowerSet}(\mathbf{AP}) = \{ \{ \}, \{\text{wait}\}, \{\text{crit}\}, \{\text{wait, crit}\} \}$$

Property: Process enters critical section infinitely often

$$((\{ \} + \{\textbf{wait}\})^* (\{\textbf{crit}\} + \{\textbf{wait, crit}\}))^\omega$$

ω-regular properties

ω-regular expressions

Next goal: Find algorithms to model-check ω-regular properties