Unit-10: Algorithms for CTL

B. Srivathsan

Chennai Mathematical Institute

NPTEL-course

July - November 2015

Module 1:
Adequate CTL formulae

Recap of CTL

State formulae

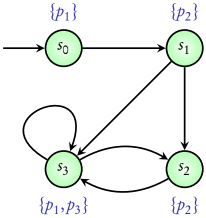
$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

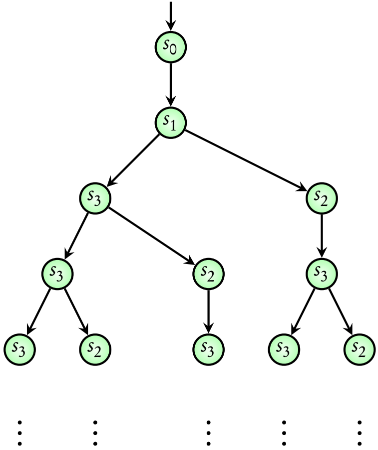
$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

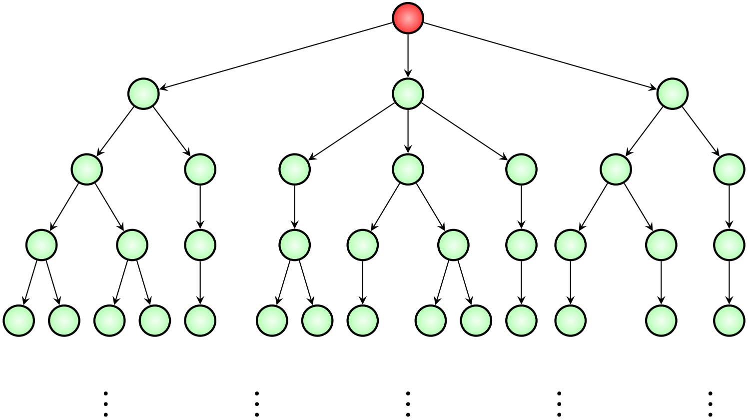
$$\alpha := X\,\phi_1 \mid \phi_1\,U\,\phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

Transition system satisfies CTL state formula ϕ if its **computation tree** satisfies ϕ

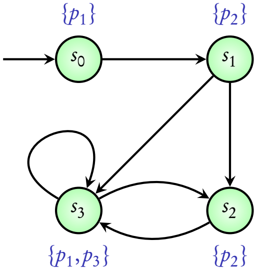


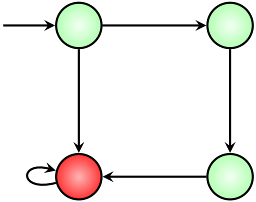


A **tree** satisfies CTL state formula ϕ if its **root** satisfies ϕ

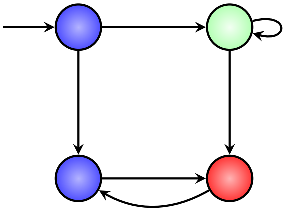


A **state** s in a transition system satisfies a CTL formula ϕ if the computation tree **starting at** s satisfies ϕ

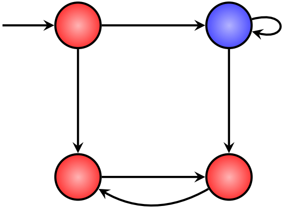




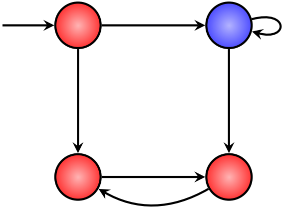
Above transition system satisfies **E X** *red*



Above transition system satisfies **E** *blue* **U** *red*



Above transition system satisfies **E G** *red*



Above transition system satisfies **E G** *red*

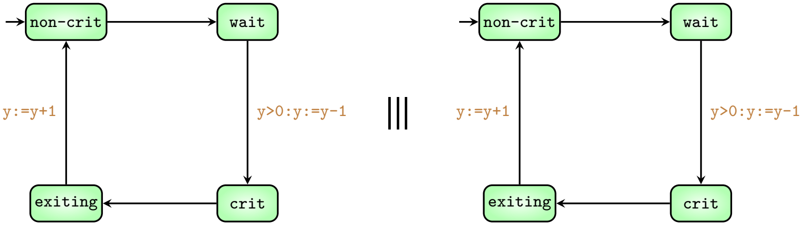
It does not satisfy **A F** *blue*

Mutual exclusion

Atomic propositions AP $= \{ p_1, p_2, p_3, p_4 \}$

p_1: `pr1.location=crit` p_2: `pr1.location=wait`

p_3: `pr2.location=crit` p_4: `pr2.location=wait`



Above system satisfies $\mathbf{A}\ \mathbf{G} \neg (p_1 \wedge p_3)$

Goal of this unit

Design an **algorithm**:

INPUT: A **transition system** M and a **CTL formula** ϕ

OUTPUT: Does M satisfy ϕ?

Goal of this unit

Design an **algorithm**:

INPUT: A **transition system** M and a **CTL formula** ϕ

OUTPUT: Does M satisfy ϕ?

We will answer a more general question:

Given M and ϕ, find all the states of M that satisfy ϕ

First step

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

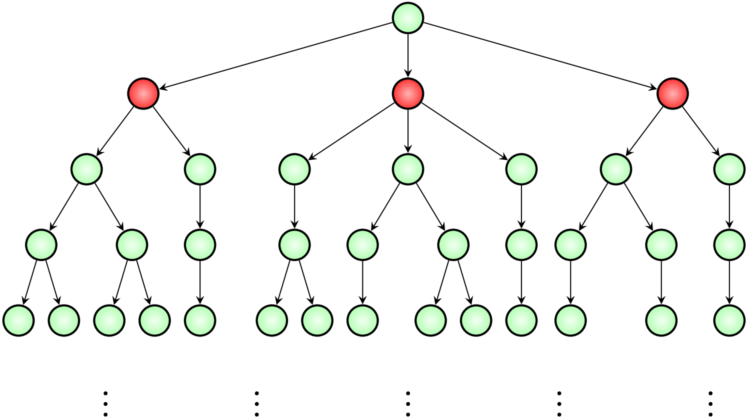
$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

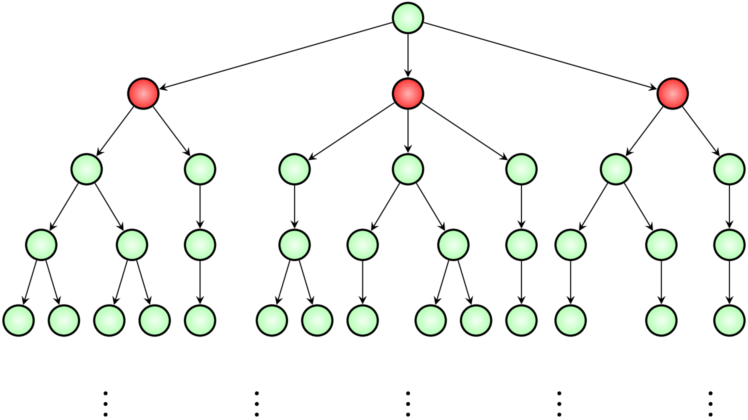
$$\alpha := X\,\phi_1 \mid \phi_1\,U\,\phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

Rewrite **A** in terms of **E**

A X (*red*) equivalent to $\neg$ **E X** ($\neg$ *red*)



A X (*red*) equivalent to ¬ **E X** (¬ *red*)



$$\mathbf{A}\,\mathbf{X}\,\phi \quad \equiv \quad \neg\,\mathbf{E}\,\mathbf{X}\,\neg\,\phi$$

Can we rewrite $\mathbf{A}\,(\phi\;\mathbf{U}\;\psi)$ as $\neg\,\mathbf{E}\,\neg\,(\phi\;\mathbf{U}\;\psi)$?

Can we rewrite $\mathbf{A}\,(\phi\ \mathbf{U}\ \psi)$ as $\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$?

No: $\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$ is **not a CTL formula**

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

$$\alpha := X\,\phi_1 \mid \phi_1\, U\, \phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

Can we rewrite $\mathbf{A}\,(\phi\ \mathbf{U}\ \psi)$ as $\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$?

No: $\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$ is **not a CTL formula**

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

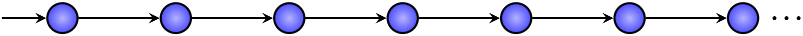
$$\alpha := X\,\phi_1 \mid \phi_1\,U\,\phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

CTL **does not allow** negation of path formula!

Coming next: Rewrite **A U** in terms of **E U** and **E G**

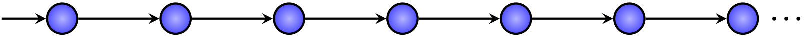
$\neg\,(\textit{blue}\ \mathbf{U}\ \textit{red})$

$\neg\,(\mathit{blue}\ \mathbf{U}\ \mathit{red})$



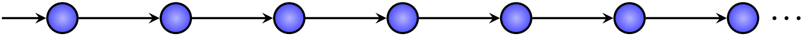
$\neg$ (*blue* **U** *red*)

G $\neg$ *red*



$\neg$ (*blue* **U** *red*)

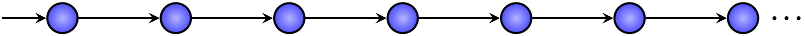
G $\neg$ *red*



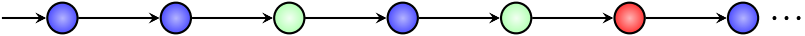
or

$\neg$ (*blue* **U** *red*)

G $\neg$ *red*

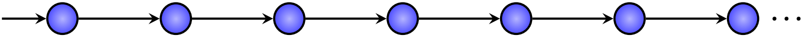


or

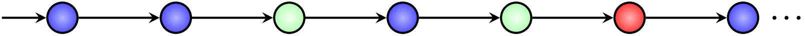


$$\neg\,(\mathit{blue}\ \mathbf{U}\ \mathit{red})$$

$$\mathbf{G}\,\neg\,\mathit{red}$$

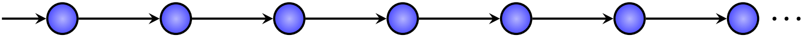


or $(\neg\,\mathit{red})\ \mathbf{U}\ (\neg\,\mathit{blue}\ \wedge\ \neg\,\mathit{red})$

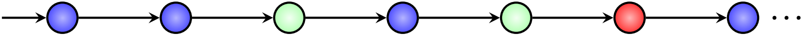


$\neg\,(\mathit{blue}\ \mathbf{U}\ \mathit{red})$

$\mathbf{G}\,\neg\,\mathit{red}$



or $(\neg\,\mathit{red})\ \mathbf{U}\ (\neg\,\mathit{blue}\ \wedge\ \neg\,\mathit{red}\,)$



$$\neg\,(\phi\ \mathbf{U}\ \psi) \quad \equiv \quad \mathbf{G}\,\neg\,\psi\ \vee\ (\,\neg\psi\ \mathbf{U}\,(\neg\phi\ \wedge\ \neg\psi\,)\,)$$

$\mathbf{A}\,(\phi\;\mathbf{U}\;\psi)$

$$\mathbf{A}\,(\phi\;\mathbf{U}\;\psi)$$

$$\equiv$$

$$\neg\,\mathbf{E}\,\neg\,(\phi\;\mathbf{U}\;\psi)$$

$$\mathbf{A}\,(\phi\ \mathbf{U}\ \psi)$$

$$\equiv$$

$$\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$$

(Not a CTL formula)

$$\mathbf{A}\,(\phi\ \mathbf{U}\ \psi)$$

$$\equiv$$

$$\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$$

(Not a CTL formula)

$$\equiv$$

$$\neg\,(\ \mathbf{E}\,\mathbf{G}\,\neg\,\psi\ \ \vee\ \ \mathbf{E}\,(\neg\psi\ \mathbf{U}\,(\neg\psi\ \wedge\ \neg\,\phi))\ )$$

$$\mathbf{A}\,(\phi\ \mathbf{U}\ \psi)$$

$$\equiv$$

$$\neg\,\mathbf{E}\,\neg\,(\phi\ \mathbf{U}\ \psi)$$

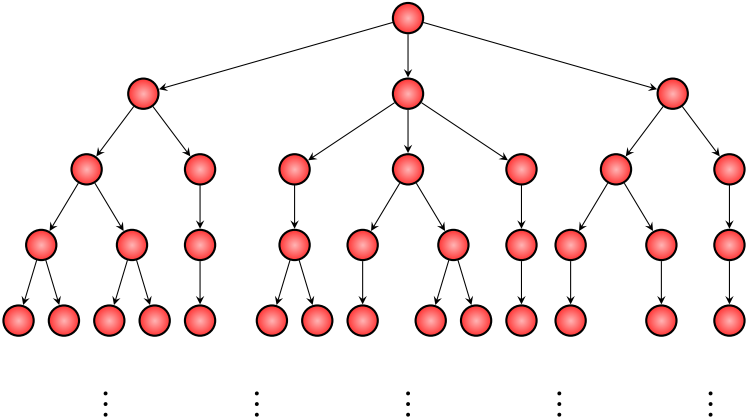
(Not a CTL formula)

$$\equiv$$

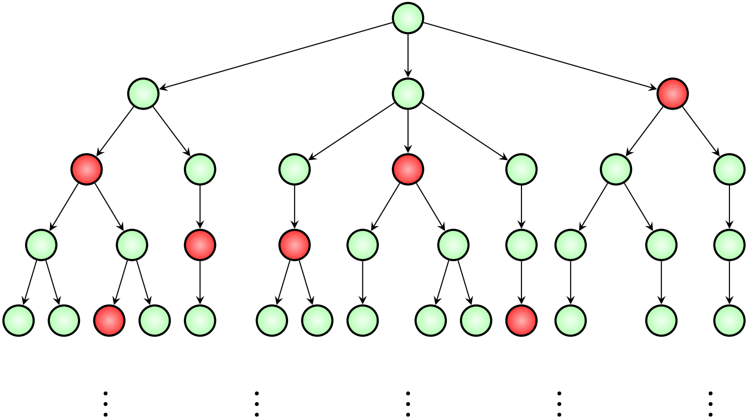
$$\neg\,(\ \mathbf{E}\,\mathbf{G}\,\neg\,\psi\ \ \vee\ \ \mathbf{E}\,(\neg\psi\ \mathbf{U}\,(\neg\psi\ \wedge\ \neg\,\phi))\ )$$

(A CTL formula!)

A G (*red*) equivalent to ¬ **E F** (¬ *red*)



A F (*red*) equivalent to ¬ **E G** (¬ *red*)



First step

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

$$\alpha := X\,\phi_1 \mid \phi_1\,U\,\phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

Rewrite **A** in terms of **E**

First step

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid E\,\alpha \mid A\,\alpha$$

$p_i \in AP$ ϕ_1, ϕ_2 : State formulae α : Path formula

Path formulae

$$\alpha := X\,\phi_1 \mid \phi_1\, U\, \phi_2 \mid F\,\phi_1 \mid G\,\phi_1$$

Rewrite **A** in terms of **E** **Done!**

All CTL formulas can be written in terms of

$\mathbf{E}\,\mathbf{X}$, $\mathbf{E}\,\mathbf{U}$, $\mathbf{E}\,\mathbf{G}$ and $\mathbf{E}\,\mathbf{F}$

All CTL formulas can be written in terms of

$\mathbf{E}\,\mathbf{X}$, $\mathbf{E}\,\mathbf{U}$, $\mathbf{E}\,\mathbf{G}$ and $\mathbf{E}\,\mathbf{F}$

Moreover $\mathbf{E}\,\mathbf{F}\ \phi \quad \equiv \quad \mathbf{E}\ (\text{true}\ \mathbf{U}\ \phi)$

All CTL formulas can be written in terms of

$$\mathbf{E}\,\mathbf{X}\,,\ \ \mathbf{E}\,\mathbf{U}\,,\ \ \mathbf{E}\,\mathbf{G}\ \ \text{and}\ \ \mathbf{E}\,\mathbf{F}$$

Moreover $\mathbf{E}\,\mathbf{F}\ \phi \quad \equiv \quad \mathbf{E}\ (\,\text{true}\ \mathbf{U}\ \phi\,)$

E X, **E U** and **E G** are adequate to describe all CTL formulas

Existential Normal Form (ENF) for CTL

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid E\,X\,\phi \mid E\,(\phi_1\,U\,\phi_2) \mid E\,G\,\phi$$

$p_i \in AP$

ϕ, ϕ_1, ϕ_2 : State formulae

Existential Normal Form (ENF) for CTL

State formulae

$$\phi := \text{true} \mid p_i \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid E\,X\,\phi \mid E\,(\phi_1\,U\,\phi_2) \mid E\,G\,\phi$$

$p_i \in AP$ ϕ, ϕ_1, ϕ_2 : State formulae

Theorem

For **every** CTL formula there exists an **equivalent** CTL formula in ENF