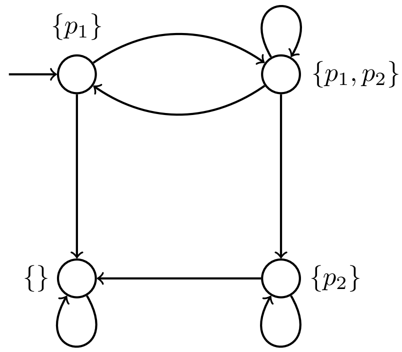
1. Suppose the set of atomic propositions is $\{p_1, p_2\}$. Consider the following transition system -



(Notation: For each state, the atoms written inside the curly braces next to the corresponding state are the atoms that are true at that state. For example, p_1 is true at the top left state (initial state), whereas p_2 is true at the bottom right state, and so on.)

Which of the following LTL formulas does this transition system satisfy?

(a) Fp_2
(b) $G(p_1 \vee p_2)$
(c) $(p_1 U p_2) \vee G(\neg p_2)$
(d) $(p_1 \wedge p_2) \rightarrow Xp_2$
(e) $G((p_1 \wedge p_2) \rightarrow Xp_2)$

Solution:

(a) System does not satisfy Fp_2. Consider the execution path $\{p_1\}\{\}^\omega$. This does not satisfy the formula Fp_2. Hence, the transition system also does not satisfy the formula Fp_2.

(b) System does not satisfy $G(p_1 \vee p_2)$. Since the bottom left state, where none of the propositions (p_1, p_2) are true, is reachable, the formula is not satisfied by this transition system.

(c) System satisfies $(p_1 U p_2) \vee G(\neg p_2)$. From the initial state the transition system can either go to the top right state, in that case $p_1 U p_2$ is true. Otherwise, the transition system goes to the bottom left state, in this case the transition system has to stay at that state forever, hence $G(\neg p_2)$ is true.

(d) System satisfies $(p_1 \wedge p_2) \rightarrow Xp_2$. Since at the initial state $(p_1 \wedge p_2)$ is false, the formula is true at the initial state. Hence, the transition system satisfies the formula.

(e) System does not satisfy $G((p_1 \wedge p_2) \rightarrow Xp_2)$. Since from the top right state the transition system can come back to the initial state in the next step, and in this state p_2 is false. Thus, the formula is not satisfied by the transition system.

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2. Suppose p, q, r are three propositional atoms.

(a) Are the two formulas $((p\ U\ q)\ U\ r)$ and $(p\ U\ (q\ U\ r))$ equivalent? That is, whichever (infinite) word satisfies the first formula would satisfy the second and vice versa?

(b) Is $(p\ U\ (q \vee r))$ equivalent to $((p\ U\ q) \vee (p\ U\ r))$?

(c) Is $((q \vee r)\ U\ p)$ equivalent to $((q\ U\ p) \vee (r\ U\ p))$?

Solution:

(a) No. The word pr^ω satisfies the second formula but does not satisfy the first.

(b) Yes. The first formula requires that $(q \vee r)$ must be true at some point, so either q or r (possibly both) must be true. Until then p must be true. So at the point when either q or r (or both) becomes true, the formulas $p\ U\ q$ or $p\ U\ r$ (or both) becomes true respectively. This satisfies the second formula. So whenever the first formula is true, the second formula must also be true. The other side can be argued in a similar way.

(c) No. qrp^ω satisfies the first formula but not the second.

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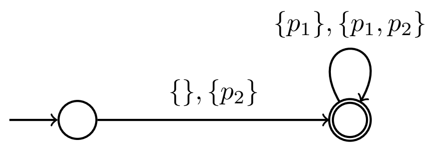
3. Suppose the set of atomic propositions is $\{p_1, p_2\}$. Consider the following NBA -



Show that the language of the above NBA is exactly the set of words satisfying the LTL formula $F(\neg p_1) \wedge XGp_1$.

Solution: Let α be a word in the language of NBA. It starts with either $\{\}$ or $\{p_2\}$. Hence α satisfies $F(\neg p_1)$. After this, each letter contains p_1. Hence α satisfies XGp_1.

Let β be a word satisfying the LTL formula $F(\neg p_1) \wedge XGp_1$. Since it satisfies XGp_1, every letter $\beta(i)$ with $i > 0$, should have p_1, and hence can be either $\{p_1\}$ or $\{p_1, p_2\}$. Since β should satisfy $F(\neg p_1)$, the only possibility for $\beta(0)$ is $\{\}$ or $\{p_2\}$.

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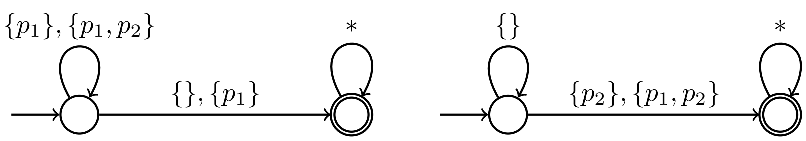
4. Draw the NBA corresponding to LTL formulas $p_1\, \boldsymbol{U}(\neg p_2)$, $(\neg p_1)\, \boldsymbol{U} p_2$.



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5. Let ϕ, ψ and χ be LTL formulas. We say two formulas are *equivalent*, written as $\phi \equiv \psi$ if they define the same language. For each of the following, prove or disprove the equivalences:

(a) $\boldsymbol{G}(\phi \wedge \psi) \equiv (\boldsymbol{G}\phi) \wedge (\boldsymbol{G}\psi)$

(b) $\boldsymbol{GFG}\phi \equiv \boldsymbol{FGF}\phi$

(c) $\boldsymbol{X}(\phi\, \boldsymbol{U} \psi) \equiv (\boldsymbol{X}\phi)\, \boldsymbol{U}(\boldsymbol{X}\psi)$

(d) $(\phi\, \boldsymbol{U}\psi)\, \boldsymbol{U}\chi \equiv \phi\, \boldsymbol{U}(\psi\, \boldsymbol{U}\chi)$

Solution:

1. True. Each letter should satisfy both ϕ and ψ.
2. Consider a word of the form $\phi(\neg\phi)\phi(\neg\phi)\ldots$. It satisfies $\boldsymbol{FGF}\phi$, but not $\boldsymbol{GFG}\phi$.
3. True. Suppose a word α satisfies $\boldsymbol{X}(\phi\, \boldsymbol{U}\psi)$. There exists an index $j > 0$ s.t. ψ is true and for all $i \in \{1, \ldots j-1\}$, we have ϕ to be true. This shows that in indices $\{1, \ldots j-2\}$ we have $\boldsymbol{X}\phi$ to be true and at $j-1$, $\boldsymbol{X}\psi$ is true. This shows α satisfies $(\boldsymbol{X}\phi)\, \boldsymbol{U}(\boldsymbol{X}\psi)$. The other direction can be similarly argued.
4. False. A word of the form $\phi(\chi)^\omega$ satisfies $\phi\, \boldsymbol{U}(\psi\, \boldsymbol{U}\chi)$ but not the other formula.
