1. Let $U = a(a+b)^*a$. Which of the following words belong to U^ω?

   i. $(abaabba)^\omega$

   ii. $(ab)^\omega$

   iii. a^ω

   Solution: i. and iii. belong to U^ω. Note that for a word to belong to U^ω, it needs to have infinitely many aa. This shows $(ab)^\omega$ does not belong to U^ω.

2. Let $U = a(a+b)^*a$. Which of the following words belong to $\lim U$?

   i. $(abaabba)^\omega$

   ii. $(ab)^\omega$

   iii. a^ω

   Solution: If a word starts with a and contains infinitely many a, then it belongs to $\lim U$. All the above words belong to $\lim U$.

3. Give an example of a regular language U such that $U^\omega \not\subseteq \lim U$.

   Solution: Pick $U = ab^*$. Consider the word $(ab)^\omega$. Clearly, $(ab)^\omega \in U^\omega$, but $(ab)^\omega \notin \lim U$.

4. Prove or disprove the following equations. Justify your answer: if you think the equation is true, provide a proof; otherwise give a specific counterexample to the equation. In the following, fix a finite alphabet Σ and assume that U and V are subsets of Σ^*. The notation V^+ stands for $V^* \setminus \epsilon$.

   i. $(U \cup V)^\omega = U^\omega \cup V^\omega$

   ii. $\lim(U \cup V) = \lim U \cup \lim V$

   iii. $U^\omega = \lim(U^+)$

   iv. $\lim(U \cdot V^+) = U \cdot V^\omega$

   Solution:

   i. Clearly, $U^\omega \subseteq (U \cup V)^\omega$ and $V^\omega \subseteq (U \cup V)^\omega$. Therefore $U^\omega \cup V^\omega \subseteq (U \cup V)^\omega$. However, it is not necessary that $(U \cup V)^\omega \subseteq U^\omega \cup V^\omega$. Take $U = \{a\}, V = \{b\}$. Then $(U \cup V)^\omega$ is the set of all infinite words formed using $\{a, b\}$. But, $U^\omega \cup V^\omega$ is the set of two ω-words $\{a^\omega, b^\omega\}$.

   ii. This statement is true.

   We first show that $\lim(U \cup V) \subseteq \lim U \cup \lim V$. Pick an ω-word $\alpha \in \lim(U \cup V)$. By definition, there exist infinitely many indices i such that $\alpha(0)\alpha(1)\cdots\alpha(i) \in U \cup V$, which implies that infinitely many of these indices come from either U or either V. This proves $\alpha \in \lim U \cup \lim V$.

   The other direction $\lim U \cup \lim V \subseteq \lim(U \cup V)$ follows directly from definition.

   iii. We show that $U^\omega \subseteq \lim(U^+)$ for every U, but there exist U such that $\lim(U^+) \not\subseteq U^\omega$.

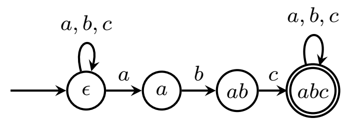
   $U^\omega \subseteq \lim(U^+)$: Pick $\alpha \in U^\omega$. Then α is of the form $u_0 u_1 u_2 \dots$ where each $u_i \in U$. Note that for each i, the word $u_0 u_1 \dots u_i \in U^+$. This gives infinitely many i such that the prefix $u_0 \dots u_i \in U^+$. Hence $\alpha \in \lim U^+$.

   $\lim(U^+) \not\subseteq U^\omega$: Take $U = ab^*$. The word $ab^\omega \in \lim(U^+)$ as $ab^i \in U$ (and hence $ab^i \in U^+$) for every i. However, $ab^\omega \notin U^\omega$.

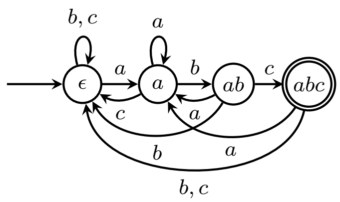
   iv. Similar to previous question. We can show that $U.V^\omega \subseteq \lim(U.V^+)$. However, $\lim(U.V^+) \not\subseteq U.V^\omega$: take $U = \{\epsilon\}$ and $V = ab^*$ and use previous answer.

5. Let $\Sigma = \{a, b, c\}$. Give Büchi automata (deterministic or non-deterministic) for the following languages:
   i. set of all ω-words where abc occurs at least once
   ii. set of all ω-words where abc occurs infinitely often
   iii. set of all ω-words where abc occurs finitely often

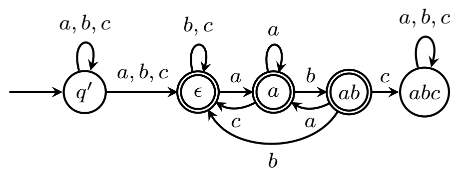
   Solution:



NBA for i.



NBA for ii.



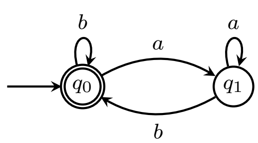
NBA for iii.

6. Let $\Sigma = \{a, b\}$. Give a Büchi automaton for the following language:

$$\{ \ \alpha \in \Sigma^\omega \mid \forall i \text{ if } \alpha(i) = a \text{ then } \exists j > i \text{ s.t. } \alpha(j) = b \ \}$$

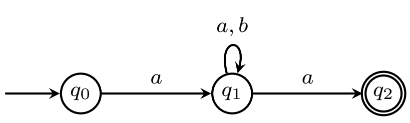
   Recall that we write α as $\alpha(0)\alpha(1)\alpha(2)\ldots$ where $\alpha(i)$ denotes the letter at the i^{th} position.

   Solution:

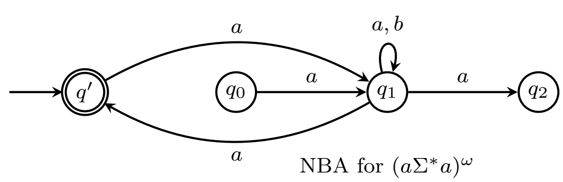


7. Suppose U is the regular language $a(a + b)^*a$. What is the NBA for U^ω?

   Solution:



NFA for $a\Sigma^*a$



NBA for $(a\Sigma^*a)^\omega$

8. Let L be an ω-language. Suppose L is of the form U^ω for some U. Does it mean that there is no deterministic Büchi automaton which can recognize L?

   Solution: No. Consider $L = \Sigma^\omega$. There is also a deterministic Büchi automaton for it.

9. Let $\mathcal{A}$ be an NFA. Let $\mathcal{B}$ be an NBA whose structure is identical to $\mathcal{A}$. Answer the following:

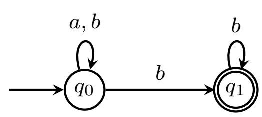
   i. Is $\lim(\mathcal{L}(\mathcal{A})) \supseteq \mathcal{L}(\mathcal{B})$?

   ii. Is $\lim(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{B})$?

   Solution:

   i. It is true that $\mathcal{L}(\mathcal{B}) \subseteq \lim(\mathcal{L}(\mathcal{A}))$. Pick a word $\alpha \in \mathcal{L}(\mathcal{B})$. It has a run $\rho : q_0 \to q_1 \to \cdots$ which visits accepting states infinitely often. Each prefix of word corresponding to $q_0 \to q_1 \to \cdots \to q_i$ such that q_i is an accepting state gives a word in $\mathcal{L}(\mathcal{A})$. This shows that $\alpha \in \lim(\mathcal{L}(\mathcal{A}))$.

   ii. No, this is not true. Consider the following automaton: By looking at it as NFA $\mathcal{A}$, we get that



   $\mathcal{L}(\mathcal{A})$ as the set of all words ending in b. By looking at it as NBA $\mathcal{B}$, we get $\mathcal{L}(\mathcal{B})$ to be the set of all ω-words that contain a only finitely often. The word $(ab)^\omega$ belongs to $\lim(\mathcal{L}(\mathcal{A}))$, but does not belong to $\mathcal{L}(\mathcal{B})$.

10. Let UP be the set of all ω-words over $\{0, 1\}$ that are ultimately periodic. Show that UP is not ω-regular.

    Solution: Consider the complement of UP (which is the set of all words that are not ultimately periodic), denoted as UP^c. Suppose UP is ω-regular. Then UP^c is also ω-regular. But we know that every non-empty ω-regular language has an ultimately periodic word. However, UP^c does not have an ultimately periodic word. This leads to a contradiction. Hence UP is not ω-regular.