Tutorial on Mean-Payoff Games

1. Let p_1, p_2, q_1, q_2 non-zero integers with $1 \leq q_1, q_2 \leq n$.

$$\left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| = \frac{|p_1 q_2 - p_2 q_1|}{q_1 q_2}$$

When $q_1 = q_2 = q$ (say):

$$\left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| = \frac{|p_1 - p_2|}{q} \geq \frac{1}{q} \geq \frac{1}{n} \geq \frac{1}{n(n-1)}$$

When $q_1 \neq q_2$:

$$q_1 q_2 \leq n(n-1)$$

$$\therefore \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| = \frac{|p_1 q_2 - p_2 q_1|}{q_1 q_2} \geq \frac{1}{q_1 q_2} \geq \frac{1}{n(n-1)}$$

This shows that there cannot be two rationals of the required form in the $\frac{1}{n(n-1)}$ interval.

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2. We will use the result that value $v(a)$ is the same as the value in the first-cycle version of the game.

- Weight of any "first cycle"

$$= \frac{\text{Sum of weights on edges}}{\text{\# edges in the cycle}}$$

$$\in [-W, W]$$

- $v(a)$ equals the weight of some first cycle that is obtained by a pair of optimal strategies.

These two observations prove the required statements.

3. Value = 1

Value iteration algorithm: $k = 4n^3 W = 4 \times 3^3 \times 1$

$$\begin{array}{c|cccccc}
0 & 0 & 1 & 2 & & 4 \times 3^2 \\
1 & 0 & 1 & 2 & \cdots & 4 \times 3^3 \\
2 & 0 & 1 & 2 & & 4 \times 3^3 \\
 & V_0 & V_1 & V_2 & & V_k
\end{array}$$

$$V' = \frac{V_k}{k} = 1 \quad \text{for all vertices.}$$

V is the unique rational no. between:

$$1 - \frac{1}{2.3.2} \quad , \quad 1 + \frac{1}{2.3.2}$$

$$\left[1 - \frac{1}{12} \ , \ 1 + 1/12\right]$$

- Algorithm will conclude 1 as the answer.

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4. Similar as above.

$k = 4n^3$

Algorithm will run for $4n^3$ steps:

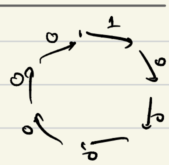
$$V' = \frac{v_k}{k} = 1$$

By similar analysis algorithm will conclude 1 as final answer.

5.

$$\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & \cdots & n & n{+}1 & \cdots & 2n \\
0 & 1 & 1 & 1 & & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & & 1 & 1 & 1 & 2 \\
\vdots & 0 & 0 & 0 & \cdots & 1 & 1 & \vdots & \cdots & \vdots & \cdots \\
 & \vdots & \vdots & 1 & & \vdots & \vdots & & & \vdots \\
0 & 0 & 1 & 1 & & 1 & 1 & 2 & & 2
\end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{n \text{ steps}} \qquad \underbrace{\qquad\qquad\qquad}_{n \text{ steps}}$$



After $4n^3$ steps. each vertex has value $4n^2$.

$$\therefore V' = \frac{V_k}{k} = \frac{4n^2}{4n^3} = \frac{1}{n}$$ Value will also be $1/n$.

6. # Vertices $= (n-1) + n + n = 3n - 1$

Value at the start vertex $= 1/_{n-1}$

↳ This comes from the cycle of length $n-1$.

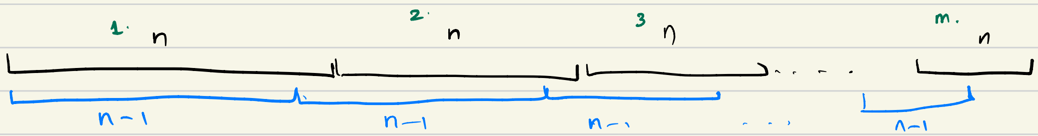
However, due to the large prefix for the n-cycle, the value due to the n-cycle is higher for some number of iterations after which the $n-1$ cycle takes over. We will find the position where this happens.

Suppose we perform $m \cdot n$ iterations.

Value due to the n-cycle $= nW + (m-1)$.

In these m steps, suppose we have unfolded the $(n-1)$ cycle m' times.

Value due to the $(n-1)$-cycle $= m'$



We have

$$m'(n-1) + c = mn \quad \text{for some } 0 \leq c < n-1$$

We want: $\quad m' \geq nW + (m-1)$

ie, $\quad \dfrac{mn - c}{n-1} \geq nW + (m-1)$

$$mn - c \geq n(n-1)W + (m-1)(n-1)$$

$$mn - c \geq n(n-1)W + mn - m - n + 1$$

$$\therefore \quad m \geq n(n-1)W - n + 1 + c$$

$$\Rightarrow \quad m \geq n(n-1)W - n + 1 \quad \text{as } c \geq 0$$

$\therefore$ No. of iterations needed to get the accurate value $= mn$

$\geq (n(n-1)W - n + 1)\, n$.

$\Omega(n^3 W)$

8\. Consider a simple cycle : $u_1 \longrightarrow u_2 \longrightarrow \cdots u_m \longrightarrow u_1$

The sum of weights in the MPG equals: $-(-n)^{p(u_1)} - (-n)^{p(u_2)} - \cdots - (-n)^{p(u_m)}$

If $p(u)$ is odd, then $-(-n)^{p(u)} > 0$

When the max priority is odd, we have one term $-(-n)^{d} = n^d$

ority.

Every term with smaller priority is between $-n^{(d-1)}$ and $-n^{(d-1)}$

There are at most n terms.

$\therefore$ The sum of weights of a cycle with max priority odd is > 0.

\- For the MPG, we can look at the first cycle version.

Any winning strategy for P1 in the Parity game will ensure that all cycles have max priority odd. This strategy will give $v(a) > 0$ in the MPG.

\- For the converse. Suppose we have a simple cycle $u_1 \rightarrow u_2 \rightarrow \cdots u_m \rightarrow u_1$

s.t. $-(-n)^{p(u_1)} - (-n)^{p(u_2)} - \cdots - (-n)^{p(u_m)} > 0$

Then the max priority should be odd.

$\therefore$ Winning strategy in MPG gives winning strategy in Parity game.