Hyper-Ackermannian Bounds for Pushdown Vector Addition Systems

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ACTS, Chennai, India, Feb 2015

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1. Pushdown Vector Addition Systems
2. Reduced Reachability Tree for Pushdown VAS
3. Worst-Case Size of the Reduced Reachability Tree
4. Conclusion

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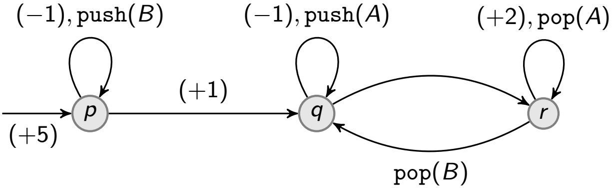
Pushdown Vector Addition Systems — Model



VAS

d (implicit) counters over $\mathbb{N}$

counter actions

- syntax: $\boldsymbol{a} \in \mathbb{Z}^d$
- semantics: $\underbrace{\boldsymbol{v}}_{\in\mathbb{N}^d} \to \underbrace{\boldsymbol{v} + \boldsymbol{a}}_{\in\mathbb{N}^d}$

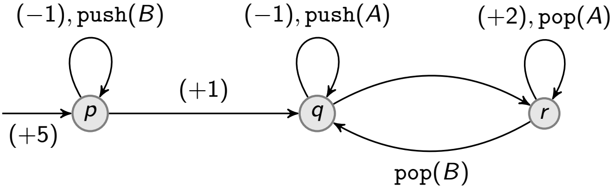
$+$

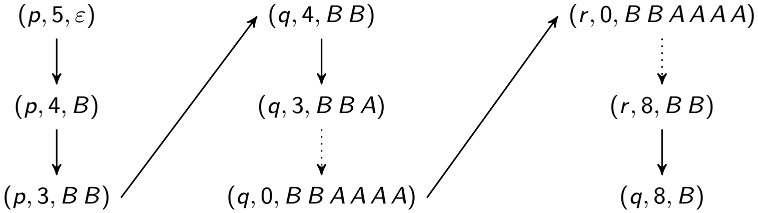
Stack

finite stack alphabet

push and pop

Pushdown Vector Addition Systems — Model

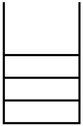




Pushdown Vector Addition Systems — Motivations

VASS
$\wr|$
VAS
$\wr|$
Petri net

$+$



➪ Richer model for the verification of concurrent systems

- Multi-threaded recursive programs
- One recursive server + unboundedly many finite-state clients

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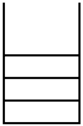

➪ Richer model for the verification of concurrent systems

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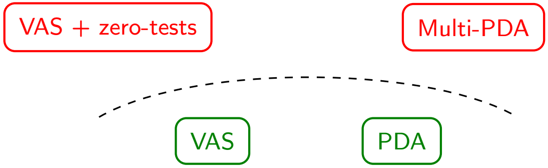
➪ Is the model too powerful?



Brief State of the Art

	Boundedness	Coverability	Reachability
VAS	ExpSpace-c[1,2]	ExpSpace-c[1,2]	Decidable[3,4]
+ full counter	Decidable[5]	Decidable	Decidable[6]
+ stack		Tower-h[7]	

[1] Lipton 1976

[2] Rackoff 1978

[3] Mayr 1981

[4] Kosaraju 1982

[5] Finkel, Sangnier 2010

[6] Reinhardt 2008

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Subclasses of pushdown VAS with decidable reachability

- Multiset pushdown systems [Sen, Viswanathan 2006]
- VAS ∩ CFL of finite index [Atig, Ganty 2011]

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Subclasses of pushdown VAS with decidable reachability

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Our Contribution

➪ Boundedness is decidable for pushdown VAS

- Reduced reachability tree: adaptation of the VAS case

➪ Worst-case complexity of the algorithm: hyper-Ackermannian

- Bound the length of bad nested sequences over $(\mathbb{N}^d, \leq)$
- Weak computation of an hyper-Ackermannian function
- Inspired from recent results on bad sequences for various wqo's
  - [Figueira, Figueira, Schmitz, Schnoebelen 2011]
  - [Schmitz, Schnoebelen 2011]
  - ...

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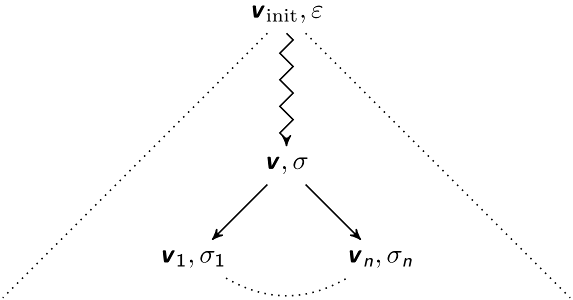
Reachability Tree of a Pushdown VAS

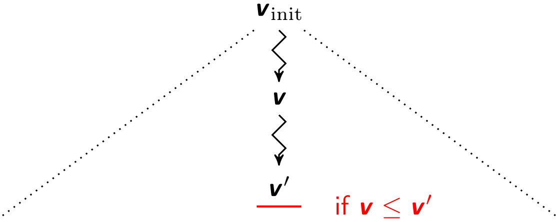


➪ Exhaustive and enumerative forward exploration from $(\boldsymbol{v}_{\text{init}}, \varepsilon)$

➪ Potentially infinite, need to truncate

Reduced Reachability Tree for VAS [Karp, Miller 1969]

Truncation rule:



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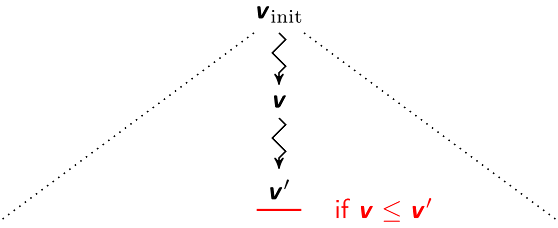

For every VAS, $\leq$ and $<$ are simulation relations

Truncation entails that

- $\boldsymbol{v}_{\text{init}} \xrightarrow{*} \boldsymbol{v} \xrightarrow{*} \boldsymbol{v}' \xrightarrow{*} \boldsymbol{v}'' \xrightarrow{*} \boldsymbol{v}''' \cdots$
- If $\boldsymbol{v} < \boldsymbol{v}'$ then $\boldsymbol{v} < \boldsymbol{v}' < \boldsymbol{v}'' < \boldsymbol{v}''' < \cdots$

RRT-based Algorithm for VAS Boundedness

Theorem ([Karp, Miller 1969])

The reduced reachability tree of a VAS $\mathcal{A}$ is finite.

Proof. König's Lemma $+$ Dickson's Lemma $\square$

Theorem ([Karp, Miller 1969])

A VAS $\mathcal{A}$ is unbounded if, and only if, its reduced reachability tree contains a leaf that is strictly larger than one of its ancestors.

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Theorem ([McAloon 1984], [Figueira et al. 2011])

The size of the reduced reachability tree of a VAS $\mathcal{A}$ is at most

- *primitive-recursive in $|\mathcal{A}|$ when the dimension d is fixed,*
- *Ackermannian in $|\mathcal{A}|$ when the dimension is part of the input.*

Tentative Simulation-Based Truncation for Pushdown VAS

Consider a run

$$(\boldsymbol{v}_{\mathrm{init}}, \varepsilon) \xrightarrow{*} (\boldsymbol{v}, \sigma) \xrightarrow{*} (\boldsymbol{v}', \sigma')$$

such that

$$\boldsymbol{v} \leq \boldsymbol{v}' \quad \text{and} \quad \sigma \leq_{\mathsf{suffix}} \sigma'$$

Then $(\boldsymbol{v}_{\mathrm{init}}, \varepsilon) \xrightarrow{*} (\boldsymbol{v}, \sigma) \xrightarrow{*} (\boldsymbol{v}', \sigma') \xrightarrow{*} (\boldsymbol{v}'', \sigma'') \xrightarrow{*} (\boldsymbol{v}''', \sigma''') \cdots$

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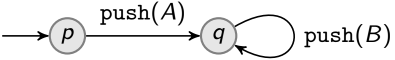
But:

$$\begin{array}{c} (p, \varepsilon) \\ \downarrow \\ (q, A) \\ \downarrow \\ (q, A\,B) \\ \downarrow \\ (q, A\,B\,B) \\ \vdots \\ \downarrow \end{array}$$



No truncation, infinite branch!

Truncation rule:

$$\boldsymbol{v}_{\mathrm{init}}, \varepsilon \rightsquigarrow \boldsymbol{v}, \sigma \overset{(\boldsymbol{u}, \rho)}{\rightsquigarrow} \underline{\boldsymbol{v}', \sigma'} \quad \text{if } \begin{cases} \boldsymbol{v} \leq \boldsymbol{v}' \\ \sigma \leq_{\text{prefix}} \rho \text{ for all } (\boldsymbol{u}, \rho) \end{cases}$$

Truncation entails that

- $(\boldsymbol{v}_{\mathrm{init}}, \varepsilon) \xrightarrow{*} (\boldsymbol{v}, \sigma) \xrightarrow{*} (\boldsymbol{v}', \sigma') \xrightarrow{*} (\boldsymbol{v}'', \sigma'') \xrightarrow{*} (\boldsymbol{v}''', \sigma''') \cdots$
- If $\boldsymbol{v} < \boldsymbol{v}'$ then $\boldsymbol{v} < \boldsymbol{v}' < \boldsymbol{v}'' < \boldsymbol{v}''' < \cdots$
- If $\sigma <_{\text{prefix}} \sigma'$ then $\sigma <_{\text{prefix}} \sigma' <_{\text{prefix}} \sigma'' <_{\text{prefix}} \sigma''' <_{\text{prefix}} \cdots$

Finiteness of the Reduced Reachability Tree

Theorem

The reduced reachability tree of a pushdown VAS $\mathcal{A}$ is finite.

Proof. By contradiction, assume that it is infinite.

The tree is finitely branching. So, by König's Lemma, there is an infinite branch

$$(\boldsymbol{v}_{\mathrm{init}}, \varepsilon) \rightarrow (\boldsymbol{v}_1, \sigma_1) \rightarrow (\boldsymbol{v}_2, \sigma_2) \cdots$$

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$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \cdots \\ & \boldsymbol{v} & & \boldsymbol{v}' \geq \boldsymbol{v} & \end{array}$$

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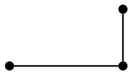

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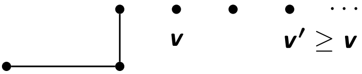

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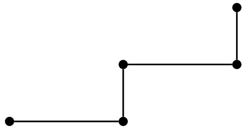

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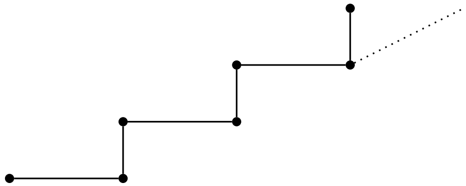

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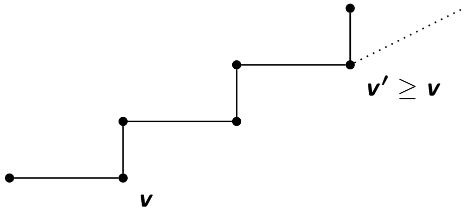

□

RRT-based Algorithm for Pushdown VAS Boundedness

Theorem

A pushdown VAS $\mathcal{A}$ is unbounded if, and only if, its reduced reachability tree contains a path

$$\underbrace{(\boldsymbol{v}, \sigma) \rightsquigarrow (\boldsymbol{v'}, \sigma')}_{\sigma \text{ remains a prefix}}$$

such that $\boldsymbol{v} \leq \boldsymbol{v}'$ and $\sigma \leq_{\mathit{prefix}} \sigma'$, and at least one of these inequalities is strict.

How big can the reduced reachability tree be?

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Fast Growing Functions

Functions $F_\alpha : \mathbb{N} \to \mathbb{N}$, indexed by ordinals $\alpha \leq \omega^\omega$

$$
\begin{aligned}
F_0(n) &= n+1, \\
F_{\alpha+1}(n) &= F_\alpha^{n+1}(n) \\
F_\lambda(n) &= F_{\lambda_n}(n) \qquad \text{if } \lambda \text{ is a limit ordinal}
\end{aligned}
$$

F_1 : linear, F_2 exponential, F_3 tower of exponentials

F_ω is an Ackermannian function

F_{ω^ω} is an hyper-Ackermannian function

Example

$$
\begin{aligned}
F_{\omega^\omega}(2) = F_{\omega^3}(2) &= F_{\omega^2.3}(2) \\
&= F_{\omega^2.2+\omega.3}(2) \\
&= F_{\omega^2.2+\omega.2+3}(2) \\
&= F_{\omega^2.2+\omega.2+2}(F_{\omega^2.2+\omega.2+2}(F_{\omega^2.2+\omega.2+2}(2)))
\end{aligned}
$$

Hyper-Ackermannian Bounds

Theorem

The height of the reduced reachability tree of a pushdown VAS $\mathcal{A}$ is at most $F_{\omega^{(d+1)}}(|\mathcal{A}|)$ where d is the dimension of $\mathcal{A}$.

Corollary

The size of the reduced reachability tree of a pushdown VAS $\mathcal{A}$ is at most

- *multiply-recursive in $|\mathcal{A}|$ when the dimension d is fixed,*
- *hyper-Ackermannian in $|\mathcal{A}|$ when the dimension is part of the input.*

Theorem

For all $n \in \mathbb{N}$, there exists a pushdown VAS $\mathcal{A}_n$, of size quadratic in n, such that the reduced reachability tree of $\mathcal{A}_n$ has at least $F_{\omega^\omega}(n)$ nodes.

Lower Bound

Weak computation of $F_{\omega^d}(n)$ by a bounded pushdown VASS $\mathcal{A}_d(n)$

- ➪ Use the stack to implement recursive calls

- ➪ Maintain n in 2 counters $\mathtt{r}$ and $\overline{\mathtt{r}}$ such that $\mathtt{r} + \overline{\mathtt{r}} = n + 1$
- ➪ Maintain $\alpha = \omega^d.\, c_d + \cdots + \omega^0.\, c_0$ in $d + 1$ counters
- ➪ Implement the inductive definition of F_α by pushdown VAS rules

Lower Bound

Weak computation of $F_{\omega^d}(n)$ by a bounded pushdown VASS $\mathcal{A}_d(n)$

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  But we cannot store the calling context α!

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➪ Implement the inductive definition of F_α by pushdown VAS rules

Trick to restore the calling context α of pending recursive calls

- Push the operations (increments and decrements) that are being performed on $\mathrm{c}_0, \ldots, \mathrm{c}_d$
- Revert them when popping

Upper Bound for VAS — Following [Figueira et al. 2011]

Each branch of the RRT is a bad sequence over $(\mathbb{N}^d, \leq)$

$$\bullet \quad \bullet \quad \underset{\boldsymbol{v}}{\bullet} \quad \underbrace{\bullet \quad \bullet \quad \bullet \quad \cdots\cdots \quad \bullet}_{\not\geq \boldsymbol{v}}$$

Bad sequences are finite, but can be arbitrarily long

A sequence $v_0, v_1, \ldots$ is n-controlled if $\|\boldsymbol{v}_i\|_\infty \leq n + i$ for all $i \geq 0$

Given $S \subseteq \mathbb{N}^d$, define $L_S(n)$ to be the maximum length of n-controlled bad sequences over S

Upper Bound for VAS — Following [Figueira et al. 2011]

Each branch of the RRT is a bad sequence over $(\mathbb{N}^d, \leq)$

$$\underset{\boldsymbol{v}}{\bullet} \qquad \underbrace{\bullet \qquad \bullet \qquad \bullet \qquad \cdots\cdots \qquad \bullet}_{\not\geq\, \boldsymbol{v}}$$

Bad sequences are finite, but can be arbitrarily long

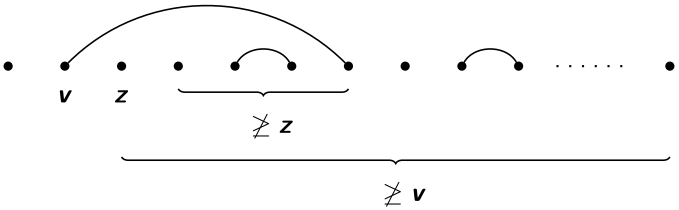
A sequence $v_0, v_1, \ldots$ is *n-controlled* if $\|\boldsymbol{v}_i\|_\infty \leq n + i$ for all $i \geq 0$

Given $S \subseteq \mathbb{N}^d$, define $L_S(n)$ to be the maximum length of n-controlled bad sequences over S

$$L_S(n) \;=\; \max_{\boldsymbol{v} \in S, \|\boldsymbol{v}\|_\infty \leq n} \; 1 \;+\; L_{S/\boldsymbol{v}}(n+1)$$

Upper Bound for Pushdown VAS

Each branch of the RRT is a bad nested sequence over $(\mathbb{N}^d, \leq)$



Define the maximum length of n-controlled bad nested sequences in the same way as non-nested ones

Upper Bound for Pushdown VAS

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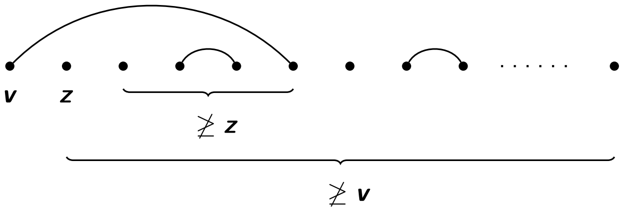

Define the maximum length of n-controlled bad nested sequences in the same way as non-nested ones

$$L_S(n) \quad = \quad \max_{\boldsymbol{v} \in S, \|\boldsymbol{v}\|_\infty \leq n} \; 1 + L_{S/\boldsymbol{v}}(n+1) + L_{S/\boldsymbol{v}}(n+1+L_{S/\boldsymbol{v}}(n+1))$$

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Summary

➪ Extension of the reduced reachability tree from VAS to pushdown VAS

- In the paper: extension to well-structured pushdown systems

➪ Boundedness and termination are decidable for pushdown VAS

➪ Hyper-Ackermannian (F_{ω^ω}) worst-case running time

- The reduced reachability tree of a pushdown VAS $\mathcal{A}$ has at most $F_{\omega^\omega}(|\mathcal{A}|)$ nodes
- This bound is tight

➪ Bounds on the reachability set when it is finite

Open Problems

➪ Complexity of the boundedness problem for pushdown VAS

- Lower bound: tower of exponentials (F_3) from [Lazić 2012]
- Upper bound: hyper-Ackermann (F_{ω^ω})

➪ Decidability of coverability / reachability for Pushdown VAS

- Coverability decidable for 1-dim VASS + stack [Submitted]
- Reachability open even for 1-dim VASS + stack

➪ Complexity of these problems for VAS + full counter

- Coverability for this model is harder than reachability for VAS

Thank You

Pushdown VAS — Syntax

Definition

A pushdown vector addition system is a triple $\langle \boldsymbol{v}_{\mathrm{init}}, \Gamma, \Delta \rangle$ where

- $\boldsymbol{v}_{\mathrm{init}} \in \mathbb{N}^d$: initial vector
- Γ : finite stack alphabet
- $\Delta \subseteq (\mathbb{Z}^d \times \mathtt{Op})$: finite set of actions, with

$$\mathtt{Op} = \{\varepsilon\} \cup \{\mathtt{push}(\gamma), \mathtt{pop}(\gamma) \mid \gamma \in \Gamma\}$$

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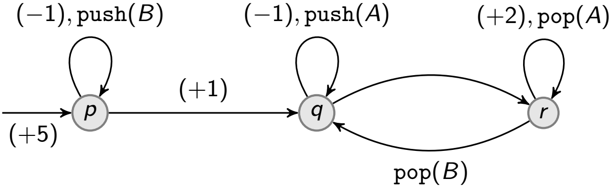


Pushdown VAS — Semantics

The semantics of a pushdown VAS $\langle \boldsymbol{v}_{\mathrm{init}}, \Gamma, \Delta \rangle$ is the transition system $\langle \mathbb{N}^d \times \Gamma^*, (\boldsymbol{v}_{\mathrm{init}}, \varepsilon), \rightarrow \rangle$ whose transition relation $\rightarrow$ is given by

$$\frac{(\boldsymbol{a}, \varepsilon) \in \Delta \;\wedge\; \boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{a} \geq \mathbf{0}}{(\boldsymbol{v}, \sigma) \rightarrow (\boldsymbol{v}', \sigma)}$$

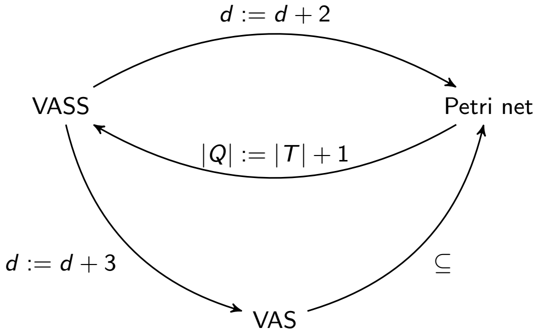
$$\frac{(\boldsymbol{a}, \mathrm{push}(\gamma)) \in \Delta \;\wedge\; \boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{a} \geq \mathbf{0}}{(\boldsymbol{v}, \sigma) \rightarrow (\boldsymbol{v}', \sigma \cdot \gamma)}$$

$$\frac{(\boldsymbol{a}, \mathrm{pop}(\gamma)) \in \Delta \;\wedge\; \boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{a} \geq \mathbf{0}}{(\boldsymbol{v}, \sigma \cdot \gamma) \rightarrow (\boldsymbol{v}', \sigma)}$$

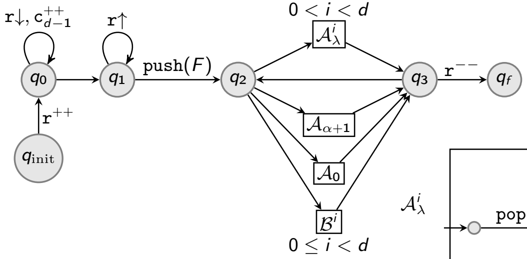
VASs $\simeq$ Petri nets $\simeq$ VASSs

Additional Feature of Petri nets

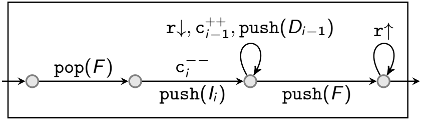
Test $\mathtt{x} \geq cst$ without modifying $\mathtt{x}$



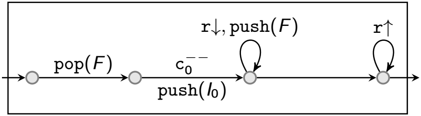
Pushdown VASS $\mathcal{A}_d(n)$ for the Lower Bound



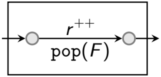
$\mathcal{A}_\lambda^i$



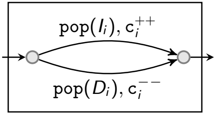
$\mathcal{A}_{\alpha+1}$



$$\mathtt{r}\!\downarrow \;=\; \mathtt{r}^{--}, \overline{\mathtt{r}}^{++}$$
$$\mathtt{r}\!\uparrow \;=\; \mathtt{r}^{++}, \overline{\mathtt{r}}^{--}$$



$\mathcal{A}_0$



$\mathcal{B}^i$