Testing Uniformity of Stationary Distribution

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Abstract. A random walk on a directed graph gives a markov chain on the vertices on the graph. An important question that arises often in the context of markov chain is whether the uniform distribution on the vertices of the graph a stationary distribution for the markov chain. Testing whether a distribution is uniform or "far" from being uniform is a well studied problem in property testing and statistics. But in most of the cases the distribution is accessed by drawing random samples. In this paper we consider the case when the distribution is the stationary distribution for the random walk on a directed graph with the uniform distribution as the initial distribution. The distribution is specified using the graph. Random samples cannot be drawn from the distribution. The distribution is accessed by querying the structure of the graph. In this paper we consider the case where the underlying undirected graph structure is known in advance and the orientation of the edges have to queried. This model for testing graph properties is called the orientation model (introduced by Halevy et al). We prove that testing whether the stationary distribution is uniform can be reduced to testing a Eulerian in graphs. Fischer et al studied the problem of testing Eulerianness in the orientation model. Using their results we obtain bounds on the query complexity for testing whether the stationary distribution is uniform.

1 Introduction

Testing of uniformity of distributions has been a very well studied problem in theoretical computer science and statistics in recent years. In most of the cases the distribution is accessed by drawing samples from the distributions. The goal usually is to understand how many samples are necessary to determine (with high probability) whether the distribution is "close" to uniform or "far" from uniform. This problem has been studied under various different assumptions on the models for drawing samples [1], [3].

However, a distribution can be specified by some other structure and drawing samples from the distribution may be infeasible. One such example is when the distribution in question is a stationary distribution of a markov chain. In this case the distribution can be specified by the transition matrix and the initial distribution. Thus, given a markov chain one can ask if the uniform distribution is a stationary distribution.

In this paper, we focus our attention to a special but very important Markov chain: random walks on a directed graph. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be an unweighted directed graph and we want to check if the stationary distribution for the markov chain (generated by the random walk on $\overrightarrow{G}$) is uniform when the initial distribution is uniform. Of-course given the directed graph it is easy to check whether uniform distribution on the vertices is a stationary distribution. But we are interested in answering the question by looking at a very small fraction of the input (in this case the directed graph). This falls into the subject of property testing where one want to look at a very small part of the input and distinguish whether an input has a certain property or is "far" from having the property. In this paper we want to look at a very small part of the directed graph and distinguish the following two cases (when the initial distribution is the uniform distribution on the vertices):

- The stationary distribution is the uniform distribution,
- The markov chain is "far" from having uniform as the stationary distribution.

Note that the above problem is exactly same as testing whether the uniform distribution on the vertices is a stationary distribution for the Markov chain corresponding to a random walk on $\overrightarrow{G}$.

The main result of this paper is the following combinatorial result which might also be of independent interest:

Theorem 1. *If $\overrightarrow{G} = (V, \overrightarrow{E})$ is a digraph such that for every vertex $v \in V$ the total degree (that is $Indegree(v) + Outdegree(v)$) is same, then the uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution (for the markov chain generated by a random walk on $\overrightarrow{G}$) if and only if the graph has the property that for every edge $(u, v) \in \overrightarrow{E}$, the outdegree of u is equal to indegree of v.*

The above result may be a bit surprising as stationary distribution of graphs is a very global property, but whether uniform is a stationary distribution depends some very local property of the graph. We also show that the property can be shown to be equivalent to Eulerianness (in case of non-bipartite graphs).

Note that the above theorem means that if $\overrightarrow{G} = (V, \overrightarrow{E})$ is a digraph such that for every vertex $v \in V$ the total degree is same, then testing whether the uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution (for the markov chain generated by a random walk on $\overrightarrow{G}$) is equivalent to testing whether the graph has the property that for every edge $(u, v) \in \overrightarrow{E}$, is the outdegree of u is equal to indegree of v.

A property of a graph that is invariant under graph isomorphism is called a graph property. Testing of graph properties has been a very hot topic in the subject of property testing (see [4], [7]). (Note that, whether the uniform distribution on the vertices is a stationary distribution for a given graph is actually a graph property.) Now when the input is a graph (as in our case) how the input is accessed is very important. Various different models have been studied in this respect, for example: dense-graph models, sparse-graph models, orientation model etc. In this paper we also study the query complexity for testing in the orientation model.

The **orientation model** for testing graph properties was introduced by Halevy *et al* [6]. This model is used for testing graph properties in directed graphs (the graphs can be multigraphs also). In this model the underlying undirected graph is known in advance. Every edge in the graph is oriented and the orientation of each edge has to be queried. Thus if the undirected graph is $G = (V, E)$ a typical query is: Is the edge $\{u, v\} \in E$ oriented as $u \to v$ or $v \to u$. The goal is to test whether a directed graph (accessed in the orientation model) has a certain property or is "far" from having this property by making as few queries as possible. Some interesting graph properties like connectedness [2] and Eulerianness [5] has been studied in this model.

We show that for both bipartite and non-bipartite graphs testing (in the orientation model) whether the uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution (for the markov chain generated by a random walk on $\overrightarrow{G}$) can be reduced to testing whether the graph is Eulerian. Using the algorithms from [5] on testing on Eulerianness in the orientation model we obtain various bounds on the query complexity for testing uniformity of the stationary distribution for the markov chain generated by a random walk on $\overrightarrow{G}$. We summarize our results in Table 1.

2 Preliminaries

2.1 Graph Notation

Throughout this paper, we will be dealing with directed graphs (possibly with multiple edges between any two vertices) in which each edge is directed only in one direction. To avoid confusion we will call them **oriented graphs**, because each edge is oriented and is not bidirectional. We will denote the oriented graph by $\overrightarrow{G} = (V, \overrightarrow{E})$ and the underlying undirected graph (that is when the direction on the edges are removed) as $G = (V, E)$. For a vertex $v \in V$ the indegree and out degree of v in $\overrightarrow{G}$ is denoted as $In_{\overrightarrow{G}}(v)$ and $Out_{\overrightarrow{G}}(v)$ respectively. An oriented graph $\overrightarrow{G} = (V, \overrightarrow{E})$ is called a degree-Δ graph if for all $v \in V$, $In_{\overrightarrow{G}}(v)$+$Out_{\overrightarrow{G}}(v) = \Delta$. In this paper we will be focusing on degree-Δ oriented graphs.

2.2 Markov Chains

A markov chain is a stochastic process on a set of states given by a transition matrix. Let S be the set of states with $|S| = n$ then the tansition matrix T is a $n \times n$ matrix with entries from positive; the rows and columns are indexed by the states; the u, v-th entry $T_{u,v}$ of the matrix denotes the probability of transiting to state v if you are at state v. Since T is stochastic so $\sum_u T_{v,u}$ must be 1. A distribution $\mu : S \to \mathbb{R}^+$ on the vertices is said to be stationary if for all vertices

$$\sum_u t_{u,v} \mu(u) = \mu(v).$$

If $\overrightarrow{G}$ is a oriented graph then a random walk on $\overrightarrow{G}$ defines a markov chain where the states are the vertices of the graph and probability that we traverse an edge is given by the quantity $p_{u,v} = 1/Out_{\overrightarrow{G}}(u)$; or in other words the transition probability $t_{u,v}$ from vertex u to vertex v is number of edges between u and v times $p_{u,v}$.

In this paper we will be only considering the markov chains that arise from random walks on oriented graphs. We will call them **random-walk-markov-chain on** $\overrightarrow{G}$, where $\overrightarrow{G}$ is an oriented graph. The uniform distribution of the vertices is a stationary distribution for this markov chain if and only if for all $v \in V$

$$\sum_{(u,v)\in\overrightarrow{E}} p_{u,v} = 1 = \sum_{(v,u)\in\overrightarrow{E}} p_{v,u}.$$

2.3 Property Testing in the Orientation Model

In this paper we are study the property testing of graph properties in the orientation model. Given an oriented graph $\overrightarrow{G} = (V, \overrightarrow{E})$ and a property P we want to test whether G satisfies the property or is "ϵ-far" from satisfying the property. In the orientation model the underlying graph $G = (V, E)$ is known in advance. Each edge in E is oriented (that is directed in exactly one direction). The orientation of the edges has to be queried. The graph is said to be ϵ-far from satisfying the property P if one has to re-orient at least ϵ fraction of the edges to make the graph have the property.

The goal is to design an ϵ-tester (that is, an algorithm) that makes queries to the orientation of the edges and does the following:

- If $\overrightarrow{G}$ satisfies the property P then the tester ACCEPTS with probability $2/3$,
- If $\overrightarrow{G}$ is ϵ-far from satisfying the property P then the tester REJECTS with probability $2/3$.

The number of edges queried by the algorithm is the query complexity of the algorithm. The goal is of course to design a tester for P with the minimum query complexity. When the graph satisfies if the tester accepts with probability 1 then the tester is called a 1-side-error tester. The usual tester is called a 2-sided error tester.

In [5] Fischer et al studied the testing whether an oriented graph $\overrightarrow{G}$ is Eulerian. They derived various upper and lower bounds for testing Eulerian-ness both in the 1-sided and 2-sided error case. They even considered various special classes of graphs like the expanders. In this paper we use their algorithms for testing whether uniform distribution is a stationary distribution for the random-walk-markov-chain.

3 Structure of Graphs with Uniform Stationary Distribution

The following Theorem is a rephrasing of the Theorem 1.

Theorem 2. *Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a degree-Δ oriented graph, Then uniform distribution is a stationary distribution iff for all $v \in V$ both $In_{\overrightarrow{G}}(v), Out_{\overrightarrow{G}}(v) \neq 0$ and for all $(u,v) \in \overrightarrow{E}$,*

$$In_{\overrightarrow{G}}(u) = Out_{\overrightarrow{G}}(v)$$

Proof: First of all note that uniform distribution is a stationary distribution for $\overrightarrow{G}$, iff for all $v \in V$

$$\sum_{u:(u,v)\in\overrightarrow{E}} p_{u,v} = 1 = \sum_{u:(v,u)\in\overrightarrow{E}} p_{v,u},$$

where $p_{u,v}$ is the transition probability from vertex u to vertex v. Since we are doing a random walk on an unweighted oriented graph so for all $(u,v) \in \overrightarrow{E}$, $p_{u,v} = \frac{1}{Out_{\overrightarrow{G}}(u)}$.

Thus, if the graph $\overrightarrow{G}$ has the property that for all $(u,v) \in \overrightarrow{E}$, $In_{\overrightarrow{G}}(u) = Out_{\overrightarrow{G}}(v)$, then note that

$$\sum_{u:(u,v)\in\overrightarrow{E}} p_{u,v} = \sum_{u:(u,v)\in\overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(u)} = \sum_{u:(u,v)\in\overrightarrow{E}} \frac{1}{In_{\overrightarrow{G}}(v)} = 1,$$

the last equality holds because the summation is over all the edges entering v (which is non-empty) and thus has $In_{\overrightarrow{G}}(v)$ number of items in the summation. Similarly,

$$\sum_{u:(v,u)\in\overrightarrow{E}} p_{v,u} = \sum_{u:(v,u)\in\overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(v)} = 1.$$

Thus, if the graph $\overrightarrow{G}$ has the property that for all $(u,v) \in \overrightarrow{E}$, $In_{\overrightarrow{G}}(u) = Out_{\overrightarrow{G}}(v)$, then uniform distribution is a stationary distribution for the markov chain.

Now let us prove the other direction, that is, let us assume that uniform distribution is a stationary distribution for the markov chain. Note that, if uniform distribution is a stationary distribution then there is a path from u to v if and only if u and v are in the same strongly connected component of $\overrightarrow{G}$. This is because, a stationary distribution is uniform if and only if for every cut $C = V_1 \cup V_2$ where $V_2 = (V \backslash V_1)$ we have

$$\sum_{(u,v)\in\overrightarrow{E},\text{ and } u\in V_1, v\in V_2} p_{u,v} = \sum_{(u,v)\in\overrightarrow{E},\text{ and } u\in V_2, v\in V_1} p_{u,v}.$$

Thus, in other words, stationary distribution is uniform implies every connected component in the undirected graph is strongly connected in the directed graph.

Let $v_0, v_1, v_2, \ldots, v_t$ be a sequence of vertices such that the following two conditions are satisfied:

- For all $i \geq 0$, $(v_{i+1}, v_i) \in \overrightarrow{E}$
- For all $i \geq 0$, $Out_{\overrightarrow{G}}(v_{2i+1}) = \min\left\{Out_{\overrightarrow{G}}(w) \ : \ (w, v_{2i}) \in \overrightarrow{E}\right\}$
- For all $i > 0$, $Out_{\overrightarrow{G}}(v_{2i}) = \max\left\{Out_{\overrightarrow{G}}(w) \ : \ (w, v_{2i-1}) \in \overrightarrow{E}\right\}$

We call such a sequence "degree-alternating" sequence of vertices.

Claim 1. Let $\{v_i\}$ be a "degree-alternating" sequence of vertices. If we define a new sequence $\{D\}$ of positive integers as follows:

- For all $k \geq 0$, d_{2k}=$In_{\overrightarrow{G}}(v_{2k})$
- For all $k \geq 0$, d_{2k+1}=$Out_{\overrightarrow{G}}(v_{2k+1})$,

then this sequence of positive integers in a non-increasing sequence.

Moreover if v_i and v_{i+1} are two consecutive vertices in the sequence such that $In_{\overrightarrow{G}}(v_{i+1}) \neq Out_{\overrightarrow{G}}(v_i)$ then the $d_{i+1} < d_i$.

Using this claim, we can finish the proof of Theorem 2. Let there be one vertex $w \in V$ such that $In_{\overrightarrow{G}}(w) \neq Out_{\overrightarrow{G}}(u)$ for some edge $(u, w) \in \overrightarrow{E}$. Let w' be the vertex such that $(w', w) \in \overrightarrow{E}$ and

$$Out_{\overrightarrow{G}}(w') = \min\{Out_{\overrightarrow{G}}(u) : (u, w) \in \overrightarrow{E}\}.$$

Now since we have already argued that in the graph every connected component has to be strongly connected so we can create an infinite sequence of vertices such that w and w' appears consecutively in the sequence infinitely often. Now by Claim 1, it means that the sequence $\{D\}$ is a non-increasing sequence that decreases infinitely many times. But this cannot happen as all the numbers in the sequence $\{D\}$ represents indegree or outdegree of vertices and hence is always a finite integer and can never be negative. Thus, if one vertex $w \in V$ such that $In_{\overrightarrow{G}}(w) \neq Out_{\overrightarrow{G}}(u)$ for some edge $(u, w) \in \overrightarrow{E}$ then we hit a contradiction.

So for all edges $(u, v) \in \overrightarrow{E}$

$$In_{\overrightarrow{G}}(v) = Out_{\overrightarrow{G}}(u).$$

□

Proof of Claim 1: This proof uses the fact that since a stationary distribution is uniform this markov chain so for all vertices v

$$\sum_{(u,v)\in\overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(u)} = 1 = \sum_{(v,w)\in\overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(v)}.$$

Let us first prove that in the sequence $\{D\}$, $d_{2i} \geq d_{2i+1}$.
Now since $Out_{\overrightarrow{G}}(v_{2i+1}) = \min\left\{Out_{\overrightarrow{G}}(w) \ : \ (w, v_{2i}) \in \overrightarrow{E}\right\}$ so

$$1 = \sum_{(u,v_{2i})\in\overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(u)} \leq \frac{In_{\overrightarrow{G}}(v_{2i})}{Out_{\overrightarrow{G}}(v_{2i+1})}, \tag{1}$$

and hence we have $In_{\overrightarrow{G}}(v_{2i}) \geq Out_{\overrightarrow{G}}(v_{2i+1})$ which by definition gives $d_{2i} \geq d_{2i+1}$.

Now let us prove that in the sequence $\{D\}$, $d_{2i-1} \geq d_{2i}$. By definition this is same as proving $Out_{\overrightarrow{G}}(v_{2i-1}) \geq In_{\overrightarrow{G}}(v_{2i})$. Since, we have assumed that the graph is a degree-Δ graph, proving $d_{2i-1} \geq d_{2i}$ is same as proving $In_{\overrightarrow{G}}(v_{2i-1}) \leq Out_{\overrightarrow{G}}(v_{2i})$.

Now just like in the other case since

$$Out_{\overrightarrow{G}}(v_{2i}) = \max\left\{Out_{\overrightarrow{G}}(w) \ : \ (w, v_{2i-1}) \in \overrightarrow{E}\right\}$$

and hence

$$1 = \sum_{(u,v_{2i-1}) \in \overrightarrow{E}} \frac{1}{Out_{\overrightarrow{G}}(u)} \geq \frac{In_{\overrightarrow{G}}(v_{2i-1})}{Out_{\overrightarrow{G}}(v_{2i})}, \tag{2}$$

and hence we have $d_{2i-1} \geq d_{2i}$.

Note that the inequalities in Equation 1 and 2 is a strict inequality if $Out_{\overrightarrow{G}}(v_{2i+1}) \neq In_{\overrightarrow{G}}(v_{2i})$ and $Out_{\overrightarrow{G}}(v_{2i-1}) \neq In_{\overrightarrow{G}}(v_{2i})$ respectively. Thus, if for any i, $Out_{\overrightarrow{G}}(v_{i+1}) \neq In_{\overrightarrow{G}}(v_i)$ then $d_{i+1} < d_i$. □

Corollary 1. *Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be an connected degree-Δ oriented graph. Then the uniform distribution of vertices is a stationary distribution for the random walk markov chain on $\overrightarrow{G}$, if and only if the following conditions apply:*

1. *If there is an odd cycle in the undirected graph $G = (V, E)$ then the graph $\overrightarrow{G}$ is Eulerian.*
2. *If G is bipartite with bipartition $V_1 \cup V_2 = V$ then $|V_1| = |V_2|$ and indegree of all vertices in one partition will be same and it will be equal to outdegree of all vertices in other partition.*

Proof: (Part 1) From Theorem 2, it follows that the uniform distribution of vertices is a stationary distribution for the random walk markov chain on $\overrightarrow{G}$ iff for every edge $(u, v) \in \overrightarrow{E}$, we have

$$In_{\overrightarrow{G}}(v) = Out_{\overrightarrow{G}}(u). \tag{3}$$

In an Eulerian graph, all vertices will have same indegree and outdegree (as $\overrightarrow{G}$ is balanced), the condition 3 is satisfied. And using Theorem 2, we conclude that uniform distribution is a stationary distribution.

Now, if $\overrightarrow{G}$ satisfies condition 3 then if there is an edge between vertex u and v (either $(u, v) \in \overrightarrow{E}$ or $(v, u) \in \overrightarrow{E}$) then $In_{\overrightarrow{G}}(v) = Out_{\overrightarrow{G}}(u)$. Thus, if there is a path of odd length between u and v (in the undirected graph) then $In_{\overrightarrow{G}}(v) = Out_{\overrightarrow{G}}(u)$. Thus, if the undirected graph has an odd cycle passing through v then $In_{\overrightarrow{G}}(v) = Out_{\overrightarrow{G}}(v)$. Also, note that if for a vertex indegree is equal to outdegree then for all the vertices in the connected component indegree is equal to outdegree. Thus implies that every connected component that has an odd cycle must be eulerian.

(Part 2) If we start traversing from u (a vertex from left partition of bipartite graph $\overrightarrow{G}$) on underlying undirected graph G of $\overrightarrow{G}$ then all vertices which are at even length distance from u will fall on left partition of $\overrightarrow{G}$ and will have indegree $In_{\overrightarrow{G}}(v)$. Now proof of Corollary follows by equation 3. □

4 Testing for Uniformity of Stationary Distribution in the Orientation Model

Given a degree-Δ oriented graph $\overrightarrow{G} = (V, \overrightarrow{E})$ we say that the graph has the property $\mathcal{P}$ if for all $(u, v) \in \overrightarrow{E}$ we have

$$In_{\overrightarrow{G}}(u) = Out_{\overrightarrow{G}}(v)$$

In Theorem 2, we proved that uniform distribution is a stationary distribution for the random walk on a degree-Δ oriented graph $\overrightarrow{G}$ iff the graph $\overrightarrow{G}$ satisfies the property $\mathcal{P}$. Thus given a degree-Δoriented graph $\overrightarrow{G}$, testing whether uniform distribution is a stationary distribution for the random walk on $\overrightarrow{G}$ is same as testing if the graph has the property $\mathcal{P}$. Since we are interested in the property testing version of this problem we want to distinguish whether the graph $\overrightarrow{G}$ has the property $\mathcal{P}$ or is "far" from having the property.

In the orientation model the undirected graph is known in advance. The orientation of each edge has to be queried. Note that uniform distribution is a stationary distribution for the random walk on the whole graph $\overrightarrow{G}$ iff that uniform distribution is a stationary distribution for the random walk on every connected component of $\overrightarrow{G}$. Since the undirected graph is known in advance so we have the connected components and hence we will test each component separately. Note that if the graph $\overrightarrow{G}$ is ϵ-far from satisfying the property that uniform distribution is a stationary distribution for the random walk on the graph then there is at least one connected component of $\overrightarrow{G}$ that is also ϵ-far from satisfying the property $\mathcal{P}$. Thus we can do the testing connected-component wise. Thus without loss of generality we can assume that the graph $\overrightarrow{G}$ is connected.

So let us assume $\overrightarrow{G} = (V, \overrightarrow{E})$ is a connected. From Corollary 1, if $\overrightarrow{G}$ is non-bipartite then we have to test whether $\overrightarrow{G}$ is Eulerian. Since we can determine whether a connected component is bipartite or not just by looking at the underlying undirected graph, so $\overrightarrow{G}$ is non-bipartite we use the testing algorithm from [5] to test for Eulerianness.

Now let $\overrightarrow{G}$ be bipartite. Then since $\overrightarrow{G}$ is connected, so there is a unique partition of the vertex set of $\overrightarrow{G}$ that makes it bipartite and that partition can be found in the preprocessing phase just by looking at the underlying graph. Let the bipartition be V_L and V_R. Now by Corollary 1, if $|V_L| \neq |V_R|$ then the graph surely does not have have uniform distribution as a stationary distribution for the random walk on $\overrightarrow{G}$. And if $|V_L| = |V_R|$ must also have the property that outdegree of all vertices in V_L must be equal to the indegree of all vertices in V_R and vice versa.

Let v be a vertex in V_L and let $In_{\overrightarrow{G}}(v) = k_1$ and $Out_{\overrightarrow{G}}(v) = k_2$. The consider any bipartite directed graph $\overrightarrow{G^*} = (V, \overrightarrow{E^*})$ with bipartition V_L and V_R that satisfies the following conditions:

- The underlying undirected graph for $\overrightarrow{G}$ is exactly same as the underlying undirected graph for $\overrightarrow{G^*}$, and
- For all $v \in V_L$, $In_{\overrightarrow{G^*}}(v) = k_2$ and $Out_{\overrightarrow{G^*}}(v) = k_1$, and
- For all $v \in V_R$, having $In_{\overrightarrow{G^*}}(v) = k_1$ and $Out_{\overrightarrow{G^*}}(v) = k_2$.

Note that, if such graphs does not exist that means that $\overrightarrow{G}$ also cannot have the property $\mathcal{P}$. If such graph exists then consider the graph $\overrightarrow{G^{\oplus}} = (V, E + \overrightarrow{E^*})$ obtained by superimposing $\overrightarrow{G}$ and $\overrightarrow{G^*}$. Clearly, if $\overrightarrow{G}$ has the property $\mathcal{P}$ then $\overrightarrow{G^{\oplus}}$ is Eulerian. And farness from property $\mathcal{P}$ is also true by following lemma:

Lemma 1. *If $\overrightarrow{G}$ is ϵ-far from having property $\mathcal{P}$ then $\overrightarrow{G^{\oplus}}$ is $\frac{\epsilon}{2}$-far from being Eulerian.*

Proof: Let $\overrightarrow{H}$ be Eulerian graph which is the closest to $\overrightarrow{G^{\oplus}}$. Now, look at edges of $\overrightarrow{G^{\oplus}}$ that were flipped in order to obtain $\overrightarrow{H}$.

Since the underlying undirected graph for $\overrightarrow{G}$ and $\overrightarrow{G^*}$ is exactly same so there is a one to one correspondence between the edges in $\overrightarrow{E}$ and $\overrightarrow{E^*}$.

Suppose an edge (say e) flipped belonged to $\overrightarrow{E^*}$. Then, we can re-flip this edge e and flip the corresponding edge e in E. So, we have effectively flipped the same number of edges. By performing these operations on flipped edges of $\overrightarrow{E^*}$, we have obtained a new graph which has same number of flipped edges as $\overrightarrow{H^{\oplus}}$ and all the flipped edges in $\overrightarrow{G^{\oplus}}$ belong to $\overrightarrow{E}$.

So, if the graph $\overrightarrow{G^{\oplus}}$ is not $\frac{\epsilon}{2}$-far from being Eulerian, then, $\overrightarrow{G}$ is not ϵ far from having property $\mathcal{P}$, which is a contradiction. □

So all we have to test is whether the new graph $\overrightarrow{G^{\oplus}}$ is Eulerian or $\epsilon/2$-far from being Eulerian. Note that every query to $\overrightarrow{G^{\oplus}}$ can be simulated by a single query to $\overrightarrow{G}$. Thus we can now use the Eulerian testing algorithm from [5]. The algorithm is summarize in Algorithm 1.

From [5] we obtain various bounds on the query complexity (in the orientation model) for testing whether uniform distribution is a stationary distribution for the random walk on a degree-Δ oriented graph. We summarize the bounds in Table 1.

Data: Degree-Δ Oriented Graph $\overrightarrow{G} = (V, \overrightarrow{E})$
Result: Whether $\overrightarrow{G}$ has the property $\mathcal{P}$ or is ϵ-far from having it.

if G *is non-bipartite* **then**
  Test $\overrightarrow{G}$ for Eulerianness (see [5]) and give the corresponding output.
else
  Let V_L and V_R be the bipartition for the graph G.
  Sample a vertex from V_L and query all edges incident on it. Let $In_{\overrightarrow{G}}(v) = k_1$ and $Out_{\overrightarrow{G}}(v) = k_2$.
  Construct any bipartite graph $\overrightarrow{G^*} = (V, \overrightarrow{E*})$ with bipartition (V_L, V_R) such that
  (a) all $v \in V_L$ have $In_{\overrightarrow{G^*}}(v) = k_2$ and $Out_{\overrightarrow{G^*}}(v) = k_1$. and all $v \in V_R$ have $In_{\overrightarrow{G^*}}(v) = k_1$ and $Out_{\overrightarrow{G^*}}(v) = k_2$, and
  (b) the underlying graph of $\overrightarrow{G^*}$ is exactly same as $G = (V, E)$.
  Superimpose $\overrightarrow{G^*}$ and graph $\overrightarrow{G}$, and we get a graph (say $\overrightarrow{G^{\oplus}}$=($V$, $\overrightarrow{E} + \overrightarrow{E^*}$).
  Test $\overrightarrow{G^{\oplus}}$ for Eulerianness (see [5]) and give the corresponding output.
end

Algorithm 1: Algorithm for testing property $\mathcal{P}$

	1-sided test	**2-sided test**
Graphs with large Δ	$\Delta + O(\frac{m}{\epsilon^2 \Delta})$	$\Delta + \min\left\{\tilde{O}\left(\frac{m^3}{\epsilon^6 \Delta^6}\right), \tilde{O}\left(\frac{m}{\epsilon^2 \Delta^{\frac{3}{2}}}\right)\right\}$
Bounded-degree graphs *	$\Omega\left(m^{\frac{1}{4}}\right)$	$\Omega\left(\sqrt{\frac{\log m}{\log\log m}}\right)$
α-expander	$O\left(\frac{\Delta \log\left(\frac{1}{\epsilon}\right)}{\alpha\epsilon}\right)$	$min\left\{\tilde{O}\left(\left(\frac{\log\left(\frac{1}{\epsilon}\right)}{\alpha\epsilon}\right)^3\right), \tilde{O}\left(\left(\frac{\sqrt{\Delta}\log\left(\frac{1}{\epsilon}\right)}{\alpha\epsilon}\right)\right)\right\}$

* Lower bound holds for 4-regular graph.

Table 1. Bounds on the query complexity.

5 Conclusion

We showed that testing (in the orientation model) whether the stationary distribution for the random-walk-markov-chain on $\overrightarrow{G}$ is uniform, when the initial configuration is uniform, can be reduced to testing Eulerianness.

The main motivation for studying the problem is to model testing of properties of distributions when the distributions are described in some compact form and drawing random samples are not feasible. In this paper we only considered the property whether the stationary distribution for a markov chain is uniform when the initial distribution is uniform. But we would like to answer more general questions like is the stationary distribution close to some other known distribution when the initial configuration is uniform. One can also ask the same questions when the graphs are not in the orientation model but in some other model like say dense graph or sparse graph model.

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References

1. Tugkan Batu, Lance Fortnow, Ronitt Rubinfeld, Warren D. Smith, and Patrick White. Testing that distributions are close. In *FOCS*, pages 259–269, 2000.
2. Sourav Chakraborty, Eldar Fischer, Oded Lachish, Arie Matsliah, and Ilan Newman. Testing *st* -connectivity. In *APPROX-RANDOM*, pages 380–394, 2007.
3. Sourav Chakraborty, Eldar Fischer, Arie Matsliah, and Ronald de Wolf. New results on quantum property testing. In *FSTTCS*, pages 145–156, 2010.
4. Eldar Fischer. The art of uninformed decisions: A primer to property testing. *Current Trends in Theoretical Computer Science: The Challenge of the New Century, G. Paun, G. Rozenberg and A. Salomaa (editors), World Scientific Publishing*, I:229–264, 2004.
5. Eldar Fischer, Oded Lachish, Ilan Newman, Arie Matsliah, and Orly Yahalom. On the query complexity of testing orientations for being eulerian. In *APPROX-RANDOM*, pages 402–415, 2008.
6. Shirley Halevy, Oded Lachish, Ilan Newman, and Dekel Tsur. Testing properties of constraint-graphs. In *IEEE Conference on Computational Complexity*, pages 264–277, 2007.
7. Dana Ron. Property testing (foundations and trends in machine learning). *Publishers: Now Publishers Inc*.