

Expanders: Vertex and Spectral Expansion

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Last class we saw three very powerful and useful combinatorial objects: extractors, expanders and pseudo-random generators. With the promise that we shall get back to extractors and PRGs in a while, let us go into expander graphs.

1 Two Notions of Expansion

1.1 Vertex Expansion

Last class we defined a notion of expansion. The definition we saw last time is called *vertex expansion*.

A d -regular graph G on n vertices is called an (n, d, ϵ) -vertex expander if for every subset S of size less than $n/2$, the number of edges going out of S is atleast $\epsilon d|S|$. That is, $E(S, V \setminus S) \geq \epsilon d|S|$.

Another definition is what is known as the *spectral expansion*.

1.2 Spectral Expansion

Let G be a n -vertex d -regular graph. Let A be the normalized adjacency matrix, which is $A_{ij} = \frac{1}{d}$ if (i, j) is an edge and 0 otherwise; the usual adjacency matrix divided by d .

The reason we do this is that since G is d -regular, each row has exactly d non-zero entries of value $\frac{1}{d}$ and so does each column. Therefore, each row and column sum up to 1. And further, this is a symmetric matrix.

There is a theorem in linear algebra that states the following:

Theorem 1. *Let A be a symmetric matrix with real entries. Then there exists a basis $\{v_1, v_2, \dots, v_n\}$ such that each v_i is an eigenvector with eigenvalue λ_i which is real. And further, the basis is orthonormal, that is $v_i \cdot v_j = 0$ if $i \neq j$ and $v_i \cdot v_i = 1$.*

And therefore, our normalized adjacency matrix has a set of orthonormal eigenvectors as a basis. Without loss of generality, we can assume that the eigenvalues are decreasing : $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Claim 2. $\lambda_1 = 1$.

Proof. Firstly, we need to show that 1 is in fact an eigenvalue. That is pretty straightforward to see. Consider the vector of all 1s, which is sometimes referred to as $\bar{1}$.

Note that $A\bar{1} = \bar{1}$ since each row of A adds up to 1. Therefore $\bar{1}$ is an eigenvector corresponding to the eigenvalue 1. We now need to show that all eigenvalues have to be at most 1.

Let λ be any eigenvalue and x be the eigenvector corresponding to λ . Look at the product Ax . Without loss of generality, assume that the first coordinate of x is the largest entry in x . Then we have

$$\begin{aligned} Ax &= \lambda x \\ \implies (Ax)_1 &= \lambda x_1 \end{aligned}$$

Now note that the LHS will merely be the average of some d coordinates of x and the average of d coordinates can never be larger than the maximum value. And hence, $\lambda \leq 1$. In fact one can slightly tweak the argument to show that $|\lambda| \leq 1$.

Therefore all eigenvalues lie between 1 and -1 ; and 1 is the largest eigenvalue. \square

Definition 1. *The second largest eigenvalue of a graph G is the second largest eigenvalue in magnitude of the normalized adjacency matrix A of G .*

$$\lambda_2(G) = \max \{|\lambda_2|, |\lambda_n|\}$$

With this, we can define our spectral expansion.

Definition 2. *A n -vertex d -regular graph G is called a (n, d, λ) -spectral expander if $\lambda = \lambda_2(G)$.*

The use of this notion of expansion is that it allows us to use linear algebra to analyse such graphs and their properties. We shall soon see that the two notions of expansion are essentially equivalent. To ease the notation used there, we shall use *Dirac's Notations* for vectors; it is more intuitive to understand in that setting.

2 Dirac's Notation

Dirac's notation of representing vectors, inner products, outer products is widely used in Physics and especially in the setting of quantum mechanics

where equations start looking messy with ordinary vectors and transpose etc. Dirac's notation cleans them up and it makes it a lot easier to understand.

- A normal vector v is written $|v\rangle$, pronounced as "ket v ", which is the column vector. And the row vector v^T is written as $\langle v|$ and pronounced as "bra v ."
- In this setting, note that the usual inner product is just the vector product $v^T v$ which translates to $\langle v|v\rangle$ which is "bra-ket (bracket) v ."
- The outer-product vv^t which gives an $n \times n$ matrix is written as $|v\rangle\langle v|$. This is used less often than the inner product but is very useful. For example, the adjacency matrix E can be expressed as

$$E = \sum_{(u,v) \in \text{Edges}} |u\rangle\langle v|$$

where $\langle v|$ is the characteristic vector of v that has 1 in the v -th position and 0 everywhere else.

This notation really cleans up a whole mess of arrow, or transposes or dot-products etc. We shall see the power of this notation by using it to prove the connection between vertex and spectral expansion.

3 Vertex Expansion \Leftrightarrow Spectral Expansion

The following theorem shows that the two notions of expansion are essentially the same.

Theorem 3. *Vertex expansion \Leftrightarrow Spectral Expansion*

- If G was a (n, d, ϵ) vertex expander, then G is also a (n, d, λ) spectral expander where $\lambda \leq 1 - (\epsilon^2/4)$.
- If G was a (n, d, λ) spectral expander, then G is also a (n, d, ϵ) vertex expander where $\epsilon \geq (1 - \lambda)/2$.

If ϵ was large, then λ will be small and vice-versa

We shall prove just the second part of the theorem.

Proof. Let G be a (n, d, λ) expander. Let S be any subset of vertices such that $|S| \leq n/2$. We want to estimate the number of edges going out of S . Let T be the set of vertices that are not in S . We need to estimate $E(S, T)$.

The first step is to write $E(S, T)$ as a product of vectors/matrices. Let u_S be the characteristic vector of S , the vector which has a 1 on exactly those coordinates that belong to S . In Dirac's notation, $|u_S\rangle = \sum_{i \in S} |i\rangle$ and similarly $|u_T\rangle$. Now look at the term $u_S^T A u_T$:

$$\begin{aligned}
u_S^T A u_T = \langle u_S | A | u_T \rangle &= \left(\sum_{i \in S} \langle i | \right) A \left(\sum_{j \in T} | j \rangle \right) \\
&= \frac{1}{d} \left(\sum_{i \in S} \langle i | \right) \left(\sum_{(u,v) \in E} |u\rangle \langle v| \right) \left(\sum_{j \in T} |j\rangle \right) \\
&= \frac{1}{d} \sum_{(i,j) \in E(S,T)} \langle i | i \rangle \langle j | j \rangle \quad (\text{all other terms become } 0) \\
&= \frac{1}{d} |E(S, T)|
\end{aligned}$$

Suppose $\{v_1, v_2, \dots, v_k\}$ is the orthonormal eigenbasis of A , let

$$\begin{aligned}
|u_S\rangle &= \sum_{i=1}^n \alpha_i |v_i\rangle \\
|u_T\rangle &= \sum_{i=1}^n \beta_i |v_i\rangle
\end{aligned}$$

And therefore, we can calculate $\langle u_T | A | u_S \rangle$ as

$$\begin{aligned}
\langle u_T | A | u_S \rangle &= \left(\sum_{i=1}^n \alpha_i \langle v_i | \right) \left(A \left(\sum_{j=1}^n \beta_j |v_j\rangle \right) \right) \\
&= \left(\sum_{i=1}^n \alpha_i \langle v_i | \right) \left(\sum_{j=1}^n \lambda_j \beta_j |v_j\rangle \right) \\
&= \sum_{i=1}^n \alpha_i \beta_i \lambda_i \langle v_i | v_i \rangle \quad (\text{since } i \neq j \implies \langle v_i | v_j \rangle = 0)
\end{aligned}$$

We know that $\lambda_1 = 1$ and that $v_1 = \frac{1}{\sqrt{n}}\bar{1} = \frac{1}{\sqrt{n}}\sum_{i=1}^n |i\rangle$. And further,

$$\begin{aligned}\langle u_S | v_1 \rangle &= \sum_{i=1}^n \alpha_i \langle v_i | v_1 \rangle \\ &= \alpha_1 \\ \implies \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle u_S | i \rangle &= \frac{|S|}{\sqrt{n}} = \alpha_1 \\ \text{Similarly, } \frac{|T|}{\sqrt{n}} &= \beta_1\end{aligned}$$

Therefore

$$\begin{aligned}\langle u_S | A | u_T \rangle - \frac{|S||T|}{n} &= \sum_{i=2}^n \alpha_i \beta_i \lambda_i \\ \left| \langle u_S | A | u_T \rangle - \frac{|S||T|}{n} \right| &\leq \left| \sum_{i=2}^n \alpha_i \beta_i \lambda_i \right| \\ &\leq \sum_{i=2}^n |\alpha_i| |\beta_i| |\lambda_i| \\ &\leq \lambda \sum_{i=2}^n |\alpha_i| |\beta_i| \\ &\leq \lambda \sqrt{\sum_{i=2}^n \alpha_i^2} \sqrt{\sum_{i=2}^n \beta_i^2}\end{aligned}$$

Now let us go back to the definition of $|u_S\rangle$. We know that $\|u_S\|^2 = |S| = \langle u_S | u_S \rangle$. Evaluating this as a linear combination of eigenvectors, we have

$$\begin{aligned}|S| = \langle u_S | u_S \rangle &= \left(\sum_{i=1}^n \alpha_i \langle v_i | \right) \left(\sum_{i=1}^n \alpha_i |v_i\rangle \right) = \sum_{i=1}^n \alpha_i^2 \\ \implies |S| - \alpha_1^2 &= \sum_{i=2}^n \alpha_i^2 \\ \implies \sum_{i=2}^n \alpha_i^2 &= |S| - \frac{|S|^2}{n} = |S| \left(\frac{n - |S|}{n} \right) = \frac{|S||T|}{n} \\ \text{Similarly } \sum_{i=2}^n \beta_i^2 &= \frac{|S||T|}{n}\end{aligned}$$

Hence the above equation becomes

$$\begin{aligned} \left| \langle u_S | A | u_T \rangle - \frac{|S||T|}{n} \right| &\leq \lambda \frac{|S||T|}{n} \\ \implies \left| E(S, T) - \frac{d|S||T|}{n} \right| &\leq d\lambda \frac{|S||T|}{n} \\ \implies E(S, T) &\geq \frac{d|S||T|}{n} - d\lambda \frac{|S||T|}{n} \end{aligned}$$

From the definition of vertex expansion, we have

$$\begin{aligned} \frac{E(S, T)}{d|S|} = \epsilon &\geq \frac{|T|}{n} (1 - \lambda) \\ &\geq \frac{1}{2} (1 - \lambda) \quad (\text{since } |S| \leq n/2 \text{ and therefore } |T| \geq n/2) \end{aligned}$$

Therefore, any graph with spectral expansion λ has vertex expansion of $\frac{1}{2}(1 - \lambda)$. \square

In essence, both notions of expansion are equivalent. It is easier to work with the spectral definition since it allows us to use a lot of linear algebra techniques to analyze.