Counting Regions of Deformations of the Braid arrangement

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by
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Abstract

Hyperplane arrangements in $\mathbb{R}^n$ where each hyperplane is of the form $x_i - x_j = s$ for some $1 \leq i < j \leq n$ and $s \in \mathbb{Z}$ are called deformations of the braid arrangement. The combinatorics related to these arrangements have been studied extensively. The main topics of concern in this thesis are calculating the characteristic polynomial and bijectively counting the regions for such arrangements.

We will exhibit the calculation of the characteristic polynomials of some examples using the finite field method. These calculations are based on the seminal work of C. A. Athanasiadis, who initiated the use of the finite field method to study hyperplane arrangements.

We will then exhibit some of the classical bijections that count the number of regions for specific examples of deformed braid arrangements. One significant example is the bijection between regions of the Shi arrangement and parking functions. Recently, O. Bernardi has expressed the number of regions of any deformation of the braid arrangement as a signed count of certain *boxed trees*. For certain special arrangements, he also obtains a bijection between the regions and certain trees. We will exhibit this bijection for the Catalan, Shi and Linial arrangements.

Applying the methods used by Bernardi, we then obtain similar bijections for other classes of hyperplane arrangements. Namely, the type C, D, B and BC Catalan arrangements and some linear arrangements as well. Finally, we use both the finite field method and Bernardi's method to study a particular hyperplane arrangement called the *boxed threshold* arrangement.

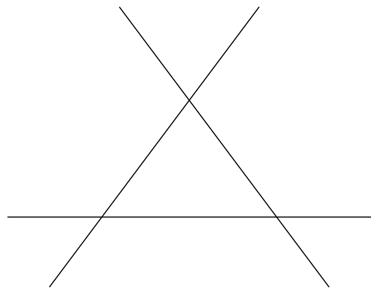
Acknowledgments

I would like to thank my supervisor, Prof. Priyavrat Deshpande, and Dr. Anurag Singh for many engaging discussions regarding this thesis. I am also very grateful to them for introducing me to the wonderful world of combinatorics.

Contents

Introduction

A (real) hyperplane arrangement is a finite set of affine hyperplanes in Euclidean space. For example, in $\mathbb{R}^2$, a hyperplane arrangement is a finite set of lines.



Looking at the above figure, we could say that the arrangement splits the plane into 7 pieces (regions). We could ask the following: How many regions does a given set of lines split the plane into? What properties of the arrangement affect this number? Making mathematical sense and generalizing such questions is what the subject of hyperplane arrangements covers.

For any hyperplane arrangement in $\mathbb{R}^n$, the number of connected components in the space obtained after removing the hyperplanes from $\mathbb{R}^n$ is called the number of regions of the arrangement. We will be focused of counting the number of regions of arrangements.

In recent years, the combinatorics associated to hyperplane arrangements has been studied extensively. The first main result in the theory of hyperplane arrangements, by T. Zaslavsky in 1975, gives the number of regions of an arrangement as the evaluation of a certain polynomial associated to the arrangement. The finite field method, developed by C. A. Athanasiadis in 1996, converts the computation of this polynomial to a counting problem. Hence a combination of these results allowed for the computation of the number of regions of several hyperplane arrangements of interest.

The search for direct proofs of these counts is an active field of research. One of the main classes of hyperplane arrangements that have been studied is the deformed braid arrangements. The classical papers on this subject covered specific examples of such arrangements using varied counting methods. In 2018, O. Bernardi developed a uniform method to count the regions of all the arrangements in this class of arrangements using trees. He in fact obtains explicit bijections for certain “well-behaved” arrangements.

In this thesis, we will illustrate how the finite field method and Zaslavsky’s theorem can be used to count regions. We will also discuss the classical bijections that count the regions of some deformed braid arrangements as well as the bijection in Bernardi’s paper.

Chapter-wise organization

Chapter 1. In Section 1.1, we define hyperplane arrangements and the various objects associated with them. In Section 1.2, we mention Zaslavsky's theorem and the finite field method. Finally in Section 1.3, we describe the main class of arrangements we will be focused on: deformations of the braid arrangement.

Chapter 2. In this chapter, we use the finite field method to obtain the characteristic polynomials for the braid (Section 2.1), Shi (Section 2.2), Linial (Section 2.3), extended Shi (Section 2.4) and extended Linial (Section 2.5) arrangements. The ideas used for counting are from [1] and [3]. Using Zaslavsky's theorem, we list the number of regions for the arrangements considered in this chapter in Table 2.1.

Chapter 3. This chapter contains some of the classical bijections used to count the regions of the braid, Catalan and Shi arrangements. The bijection for the regions of the Catalan arrangement in Section 3.2 is with certain 'sketches' and is from [5]. These sketches play an important role in Chapter 4. The bijection for the Shi regions in Section 3.3, from [4], is with parking functions, which are defined in Section 3.3.1.

Chapter 4. This chapter contains the ideas of Section 8 in [5]. Section 4.1 covers the basic definitions and terminology associated to labeled rooted trees. Section 4.2 shows how these trees are in bijection with the regions of the Catalan arrangement. The regions of the Shi and Linial arrangement are counted in Section 4.3 and Section 4.4 respectively. This is done by setting up an equivalence relation on the sketches associated to the Catalan regions. The equivalence is induced by certain valid 'moves' on the sketches. Via the bijection between sketches and trees, we also obtain a bijection between the regions of the arrangements and certain trees.

Chapter 5. In this chapter, we apply the 'sketches-moves' method of [5] to study deformations of the type C arrangement. We first consider some linear deformations in Section 5.1 and then move on to the type C Catalan arrangement in Section 5.2. The sketches and count for the type C Catalan arrangement is a modified version of that in [10]. In Section 5.2.1, we extend the definition of these sketches to cover the type C m-Catalan arrangement. Finally, in Section 5.3, we use the idea of 'moves' to bijectively count the regions of the type D, B and BC Catalan arrangement.

Chapter 6. This chapter focuses on a particular arrangement which we call the boxed threshold arrangement, first studied by Joungmin Song in [18]. The characteristic polynomial is derived using the finite field method. We then use the 'sketches-moves' method of [5] to count the number of regions of the arrangement. Extending the known bijection between regions of the threshold arrangement and threshold graphs, we obtain one between regions of the boxed threshold arrangement and certain colored threshold graphs. Most of the results in this chapter are part of [6], which is available on arXiv.

Chapter 7. In the final chapter, we discuss some potential topics for further study. These include: a better understanding of the sketches associated to the type C m-Catalan arrangement; applying the 'sketches-moves' method to other hyperplane arrangements; and finding suitable trees in bijection with the regions of the type C m-Catalan arrangement which could help prove results analogous to [5, Theorem 4.2].

This thesis contains, to the best of our knowledge, some new results. For the benefit of the reader we enumerate them here:

1. In Section 8.1 of [5], a bijection is described between regions of the m-Catalan arrangement and certain 'sketches'. The proof that the region defined by such a sketch is non-empty does not seem to work in general (see Remark 3.1). We provide an alternate proof of this fact in Proposition 3.3.
2. In [10], similar sketches are described that are in bijection with the regions of the type C Catalan arrangement. We provide a different way to describe and count these sketches in Section 5.2 (see Proposition 5.2 and Lemma 5.4). The proof that the sketches describe non-empty regions given in [10] does not seem to work in general. We provide an alternate proof in Proposition 5.3, before which we describe sketches for the type C m-Catalan arrangement (see Definition 5.2).
3. In Section 5.3, we provide bijective counts for the regions of the type D, B and BC Catalan arrangements.

Chapter 1

Preliminaries

In this chapter, we will cover the basic definitions and results related to hyperplane arrangements. The interested reader is referred to [21] for more information. We also assume reader's familiarity with the notion of posets and related terminologies. The main reference for which is [20]

1.1 Basic definitions

Definition 1.1 (Hyperplane arrangement)**.** A *hyperplane arrangement* is a finite set of affine hyperplanes in $\mathbb{F}^n$, where $\mathbb{F}$ is a field. An affine hyperplane is a translate of a codimension 1 subspace of $\mathbb{F}^n$.

We sometimes write just 'arrangement' instead of 'hyperplane arrangement'. We will be mainly focused on when $\mathbb{F} = \mathbb{R}$.

Definition 1.2 (Region)**.** A *region* of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is a connected component of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. The number of regions is denoted by $r(\mathcal{A})$.

Definition 1.3 (Rank)**.** The dimension of the span of the normals to hyperplanes in an arrangement $\mathcal{A}$ is called the rank of $\mathcal{A}$. It is denoted by $\text{rank}(\mathcal{A})$.

Definition 1.4 (Bounded regions)**.** A region of an arrangement $\mathcal{A}$ is said to be bounded if its intersection with the span of the normals of the hyperplanes in $\mathcal{A}$ is bounded. The number of bounded regions is denoted by $b(\mathcal{A})$.

Example 1.1. An arrangement in $\mathbb{R}^2$ with 7 regions is shown in Figure 1.1. It has 1 bounded region, labeled 7.

Definition 1.5 (Intersection poset)**.** The poset of non-empty intersections of hyperplanes in an arrangement $\mathcal{A}$ ordered by reverse inclusion is called its *intersection poset*. It is denoted by $L_{\mathcal{A}}$.

The ambient space of the arrangement (i.e. $\mathbb{R}^n$) is an element of the intersection poset. It is considered as the intersection of none of the hyperplanes.

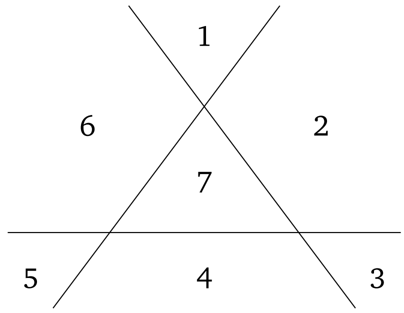


Figure 1.1: An arrangement in $\mathbb{R}^2$.

Example 1.2. Note that the lines L_1 and L_3 in the second example of Figure 1.2 do not intersect. Such empty intersections are not included in $L_{\mathcal{A}}$.

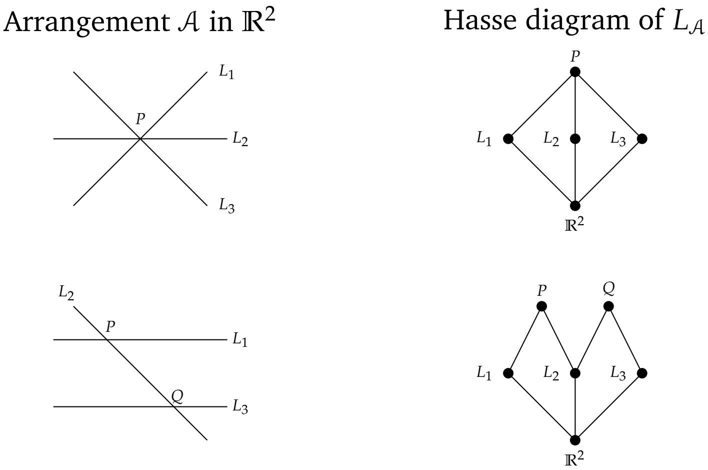


Figure 1.2: Examples of intersection poset.

Definition 1.6 (Möbius function)**.** For an arrangement $\mathcal{A}$ in $\mathbb{R}^n$, we define the *Möbius function* $\mu : L_{\mathcal{A}} \to \mathbb{Z}$ as:

$$\mu(x) = \begin{cases} 1, & \text{if } x = \mathbb{R}^n \\ -\sum\limits_{y<x} \mu(y), & \text{otherwise.} \end{cases}$$

Definition 1.7 (Characteristic polynomial)**.** The *characteristic polynomial* of an arrangement $\mathcal{A}$ is the generating function of the Möbius values of $L_{\mathcal{A}}$ weighted by dimension defined as:

$$\chi(\mathcal{A}, t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) t^{dim(x)}.$$

Example 1.3. The numbers next to elements of $L_{\mathcal{A}}$ in Figure 1.3 are their Möbius values.

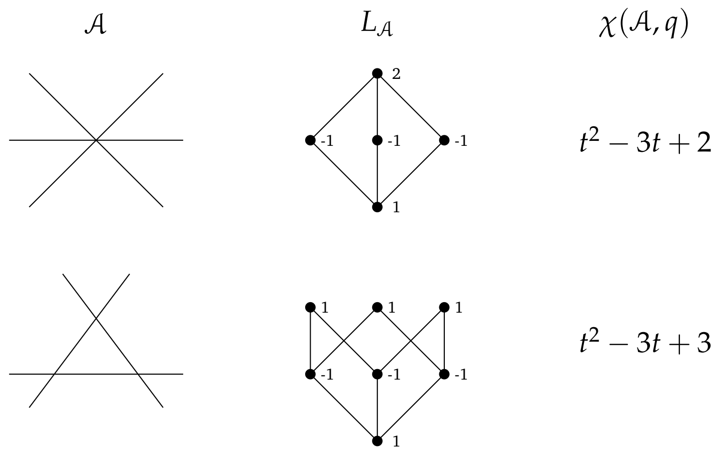


Figure 1.3: Examples of characteristic polynomial.

Definition 1.8 (Restriction)**.** The *restriction* of an arrangement $\mathcal{A}$ to some $x \in L_{\mathcal{A}}$ is the arrangement $\mathcal{A}^x$ in x with hyperplanes $\{H \cap x \mid H \in \mathcal{A},\ x \not\subseteq H\}$.

Definition 1.9 (Face)**.** A *face* of an arrangement $\mathcal{A}$ is a region of $\mathcal{A}^x$ for some $x \in L_{\mathcal{A}}$. The dimension of the face is defined as the dimension of the space x (this is the same as the dimension of its affine span).

The regions of an arrangement are themselves faces (regions of $\mathcal{A}^{\mathbb{R}^n} = \mathcal{A}$). In fact, they are the maximum-dimensional faces.

Example 1.4. The numbers inside the circles in Figure 1.4 are the dimensions of the faces.

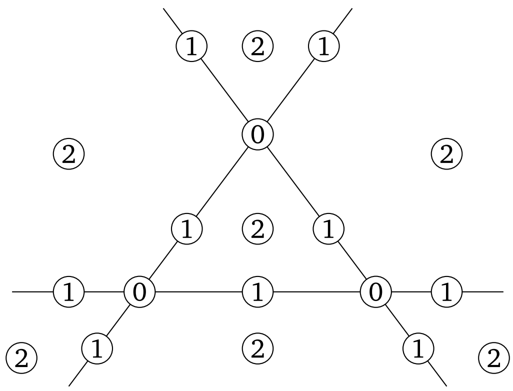


Figure 1.4: Faces of an arrangement in $\mathbb{R}^2$.

Before going further we note some results on arrangements that are consequences of basic Euclidean geometry.

- Any hyperplane in $\mathbb{R}^n$ is of the form:

$$H = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = c\}$$

for some $a_1, \dots, a_n, c \in \mathbb{R}$. We say that the *defining polynomial* of H is:

$$P_H = a_1x_1 + \cdots + a_nx_n - c.$$

It is unique up to multiplication by a nonzero scalar.

- Any region of an arrangement $\mathcal{A}$ is an intersection of sets of the form $P_H > 0$ or $P_H < 0$, where one choice is made for each $H \in \mathcal{A}$.
- Any face of $\mathcal{A}$ is an intersection of sets of the form $P_H = 0$, $P_H > 0$ or $P_H < 0$, where one choice is made for each $H \in \mathcal{A}$.

1.2 Important results

The first major theorem in the theory of arrangements was due to Zaslavsky in 1975 [23].

Theorem 1.1. *Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Then,*

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi(\mathcal{A}, -1) \\ &= \sum_{x \in L_{\mathcal{A}}} |\mu(x)| \end{aligned}$$

and

$$\begin{aligned} b(\mathcal{A}) &= (-1)^{\operatorname{rank}(\mathcal{A})} \chi(\mathcal{A}, 1) \\ &= \sum_{x \in L_{\mathcal{A}}} \mu(x). \end{aligned}$$

Remark 1.1. For both $r(\mathcal{A})$ and $b(\mathcal{A})$, the first equality is proved using Deletion-Restriction arguments. The second is obtained from the fact that the möbius function value of elements of $L_{\mathcal{A}}$ alternate in sign with respect to the dimension.

To apply combinatorial methods, we will be focused on certain "nice" arrangements.

Definition 1.10 (Rational arrangements)**.** Arrangements in $\mathbb{R}^n$ such that every hyperplane H has a defining polynomial P_H in $\mathbb{Z}[x_1, \dots, x_n]$ are called *rational arrangements*.

Even if $P_H \in \mathbb{Q}[x_1, \dots, x_n]$, we can multiply it by an integer to obtain an integer-coefficient defining polynomial for H. So the term 'rational arrangements' makes sense. Also, for such arrangements we can obtain related arrangements in vector spaces over finite fields.

Definition 1.11 (Reduction mod q)**.** If $\mathcal{A}$ is a rational arrangement in $\mathbb{R}^n$, we can obtain an arrangement in $\mathbb{Z}_q^n$, where q is a prime, by reducing mod q the defining polynomials of the hyperplanes of $\mathcal{A}$. Call this the arrangement $\mathcal{A}_q$.

We now have the vocabulary required to state a very convenient method to calculate the characteristic polynomials of rational arrangements. This method was developed by C. A. Athanasiadis in 1996 [1].

Theorem 1.2 (The finite field method). *If $\mathcal{A}$ is a rational hyperplane arrangement in $\mathbb{R}^n$, for large primes q,*

$$\chi(\mathcal{A}, q) = \#(\mathbb{Z}_q^n \setminus \bigcup_{H \in \mathcal{A}_q} H).$$

Hence the finite field method converts the problem of calculating the characteristic polynomial of rational arrangements to a counting problem. Combined with Zaslavsky's theorem, we get a nice method of getting the number of regions of rational arrangements.

1.3 Deformations of the Braid arrangement

In this subsection, we will introduce certain specific arrangements that we will be focused on.

The braid arrangement in $\mathbb{R}^n$ is the arrangement $\mathcal{A}_{\{0\}}(n)$ with the following hyperplanes:

$$\{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}.$$

A deformation of an arrangement $\mathcal{A}$ is an arrangement $\mathcal{A}'$ all of whose hyperplanes are translates of hyperplanes in $\mathcal{A}$. We will consider deformations of the braid arrangement that have hyperplanes of the form:

$$x_i - x_j = s \text{ for some } 1 \leq i < j \leq n \text{ and } s \in \mathbb{Z}.$$

The deformation of the Braid arrangement:

$$\{x_i - x_j = k \mid k \in S, \quad 1 \leq i < j \leq n\}$$

for some finite set of integers S is denoted by $\mathcal{A}_S(n)$.

The special cases of $\mathcal{A}_S(n)$ we will be focused on are enumerated in Table 1.1. For integers $a \leq b$, $[a, b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ and for an integer $n \geq 1$, $[n]$ denotes $[1, n]$.

Name	S	Name	S
Catalan	{-1,0,1}	m-Catalan	$[-m, m]$
Shi	{0,1}	m-Shi	$[-m+1, m]$
Linial	{1}	m-Linial	$[1, m]$

Table 1.1: Some special deformations of the braid arrangement.

Here m is any positive integer. Note that taking $m = 1$ gives us the original arrangement in all cases. Hence we consider the $m-$type arrangements to be extensions of the original. Table 1.2 shows the intersection of the arrangements in $\mathbb{R}^3$ with the hyperplane $x_1 + x_2 + x_3 = 0$. Since all the normals lie on this hyperplane, these figures can be extended to get the original arrangements. (The braid hyperplanes are not part of the Linial arrangement but they have been drawn as dotted lines).

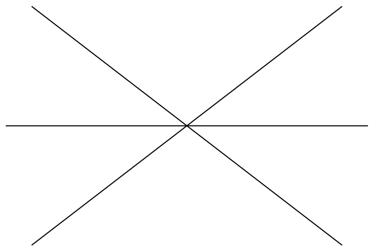
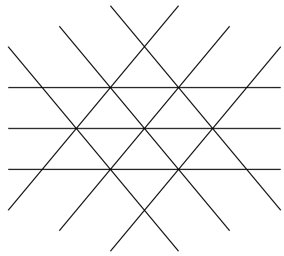
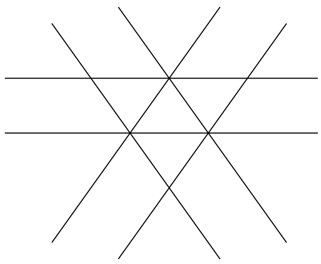
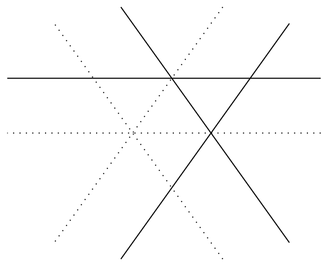
Braid	Catalan
	
Shi	Linial
	

Table 1.2: Deformed braid arrangements in $\mathbb{R}^3$.

We will now state some results for these arrangements. We will be proving most of these in subsequent sections.

- The number of regions of the braid arrangement is $n!$. The number of faces of dimension k is $k!S(n,k)$ where $S(n,k)$ is the number of ways to partition an n-element set into k blocks.

- The number of regions of the m-Catalan arrangement is $n! \times n^{th}$ m-Catalan number:

$$n! \times \frac{((m+1)n)!}{n!(mn+1)!} = \frac{((m+1)n)!}{(mn+1)!}.$$

  There is a formula for the number of k-dimensional faces of the m-Catalan arrangement for which we refer to [9].

- The number of regions of the m-Shi arrangement is the number of m-parking functions of length n, that is, $(mn+1)^{n-1}$. The formula for the number of k-dimensional faces of the m-Shi arrangement is again given in [9].

- The number of regions of the Linial arrangement is the number of alternating trees on n vertices. See [11] for the relevant definitions. No formula is known for the number of Linial faces.

There are two ways to prove such results. We could calculate the characteristic polynomial (directly or via the finite field method) and then apply Zaslavsky's theorem. Or we could use counting methods such as obtaining a bijection between the set we want to count and a set whose cardinality is known. We will use both methods to obtain most of the results mentioned above, though we will focus more on bijective methods.

Chapter 2

The finite field method

In this chapter, we will use the finite field method to compute the characteristic polynomial of some arrangements. As mentioned before, we will be focused on certain deformations of the braid arrangement. We will derive the characteristic polynomial for the Braid, extended Shi and extended Linial arrangements. The main references for this chapter are [1] and [3].

We will be using a slight extension ([3, Theorem 2.1]) of the finite field method which we state now:

Theorem 2.1 (The finite field method)**.** *If $\mathcal{A}$ is a rational hyperplane arrangement in $\mathbb{R}^n$, there exist integers m, k such that for all integers q relatively prime to m and greater than k,*

$$\chi(\mathcal{A}, q) = \#(\mathbb{Z}_q^n \setminus V_{\mathcal{A}}) \tag{2.1}$$

where $V_{\mathcal{A}}$ is the union of hyperplanes in $\mathbb{Z}_q^n$ obtained by reducing $\mathcal{A}$ mod q.

Sketch of proof. The proof follows using the Möbius inversion formula (refer [20]) once we can find integers m, k such that for all q relatively prime to m and greater than k, the dimension of the intersection of some hyperplanes of $\mathcal{A}$ is the same when reduced mod q. An intersection of hyperplanes is the solution set of a matrix equation of the form

$$Ax = b$$

where A is an $r \times n$ matrix where r is the number of hyperplanes being intersected. Since we are dealing with rational arrangements, A and b have integer entries. Using some results from linear algebra, we get that the solution space of the above equation is isomorphic to one of the form

$$A'x = b'$$

where A' is a diagonal matrix and both A' and b' have integer entries. Taking m to be the least common multiple of the entries of A' and k to be greater than absolute values of all entries of b' satisfies our requirements. $\square$

Remark 2.1. Examining the proof of the finite field method tells us that for deformations of the braid arrangement, we can take $m = 1$. So when applying the finite field method to such arrangements, (2.1) is valid for any large enough values of q, without any restriction on divisibility.

2.1 The braid arrangement

Recall that the braid arrangement in $\mathbb{R}^n$, denoted by $\mathcal{A}_{\{0\}}(n)$, is given by:

$$\{x_i - x_j = 0 \mid 1 \le i < j \le n\}.$$

Hence, by (2.1), we get that for large values of q,

$$\begin{aligned}\chi(\mathcal{A}_{\{0\}}(n), q) &= \#\{(x_1, \ldots, x_n) \in \mathbb{Z}_q^n : x_i \neq x_j \text{ for all distinct } i, j \in [n]\} \\ &= q \times (q-1) \times \cdots \times (q-(n-1)) \\ &= \prod_{i=0}^{n-1} (q-i).\end{aligned}$$

Since we have obtained a polynomial expression for large values of q, we have determined the characteristic polynomial. Using Zaslavsky's theorem we have

$$r(\mathcal{A}_{\{0\}}(n)) = n!.$$

2.2 The Shi arrangement

Recall that the Shi arrangement in $\mathbb{R}^n$ is given by:

$$\{x_i - x_j = 0, 1 \mid 1 \le i < j \le n\}.$$

We will denote it by $\mathcal{S}_n$ instead of $\mathcal{A}_{\{0,1\}}(n)$. The computation of the characteristic polynomial in this section follows the method in [3].

By (2.1), for large q,

$$\chi(\mathcal{S}_n, q) = \text{number of tuples } (x_1, \ldots, x_n) \in \mathbb{Z}_q^n \text{ such that}$$
$$x_i \neq x_j,\ x_i \neq x_j + 1 \quad \forall\, 1 \le i < j \le n$$

To count such tuples, we first need a convenient way to represent them.

We represent a tuple $(x_1, \ldots, x_n)$ in $\mathbb{Z}_q^n$ with distinct entries by a placement of q symbols: n numbers $1, \ldots, n$ and $q - n$ balls. Think of the symbols in positions $0, \ldots, q-1$ and an integer k in position i means $x_k = i$.

Example 2.1. The placement

$$5 \bigcirc \bigcirc 1\ 3 \bigcirc 2 \bigcirc 4 \bigcirc$$

represents the tuple in $\mathbb{Z}_{10}^5$ with the first coordinate being 3 since 1 is in position 3 (recall we are starting the positions with 0), the second coordinate being 6, and so on. Hence the above placement represents $(3, 6, 4, 8, 0) \in \mathbb{Z}_{10}^5$.

From now on, we consider such placements and their corresponding tuples as the same.

So, we need to count the number of such placements which satisfy $x_i \neq x_j + 1 \ \forall \ 1 \leq i < j \leq n$*, that is, we cannot have a smaller number immediately following a larger number in the placement.* (2.2)

We make our counting easier by noticing the following fact:

$$(x_1, \ldots, x_n) \in V_{\mathcal{S}_n} \Leftrightarrow (x_1 + k, \ldots, x_n + k) \in V_{\mathcal{S}_n} \ \forall \ k \in \mathbb{Z}_q$$

where $V_{\mathcal{S}_n}$ is as in (2.1).

Fix some $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}_q$. If $c =$ number of placements with $x_i = k$, for some fixed $i \in [n]$ and $k \in \mathbb{Z}_q$, by the above observation,

$$\chi(\mathcal{S}_n, q) = qc \tag{2.3}$$

Note that due to the cyclic nature of $\mathbb{Z}_q$, (2.2) also means that if the first and last symbols in a placement are both numbers, the last one must be less than the first.

To avoid having to check the above condition as well, we use (2.3) with $i = n$ and $k = 0$. So we have to count the placements with first symbol being n that satisfy (2.2). Call such placements 'valid placements'.

First we observe that in valid placements:

The numbers between a ball and the ball following it (the leftmost ball to the right of it) must be in ascending order. (2.4)

We count the valid placements by counting the number of ways there are to put in the numbers $1, \ldots, n-1$ in the diagram with n followed by $q - n$ balls shown in Figure 2.1.

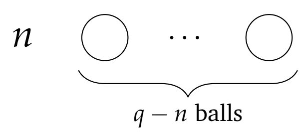


Figure 2.1: Taking $x_n = 0$.

By (2.4), we only need to specify the set of numbers between two symbols in the above diagram to specify a valid placement. Also, we cannot put any numbers from $1, \ldots, n-1$ in the space between n and the first ball.

Hence, each number from $1, \ldots, n-1$ can be put into any one of $q - n$ spaces (spaces after each ball). So we get a total of $(q-n)^{n-1}$ ways of putting in the numbers $1 \ldots, n-1$ in the diagram above to get a valid placement.

So, by (2.3),

$$\chi(\mathcal{S}_n, q) = q(q-n)^{n-1}.$$

Since our expression is a polynomial in q, we have obtained the characteristic polynomial of the Shi arrangement.

2.3 The Linial arrangement

Recall that the Linial arrangement in $\mathbb{R}^n$ is given by:

$$\{x_i - x_j = 1 \mid 1 \le i < j \le n\}.$$

We will denote it by $\mathscr{L}_n$ instead of $\mathcal{A}_{\{1\}}(n)$. The computation of the characteristic polynomial in this section follows the method in [1].

Since $\mathscr{L}_n$ does not include the $x_i - x_j = 0$ type hyperplanes, the tuples we have to count need not have distinct elements. This makes directly counting them quite cumbersome. So, we instead obtain a bijection of the tuples we need to count with a set that is easier to count.

Though we could modify the "ball-number" description of a tuple, the following might be easier to visualize:

We represent an n-tuple in $\mathbb{Z}_q^n$ by a placement of the numbers $1, \ldots, n$ into q boxes arranged cyclically. Call these "circular placements". The boxes are labeled cyclically with the elements of $\mathbb{Z}_q$. If the number i is in the box labeled k, then $x_i = k$.

Example 2.2. The diagram corresponding to $(0, 0, 5, 2, 3) \in \mathbb{Z}_7^5$ is shown in Figure 2.2.

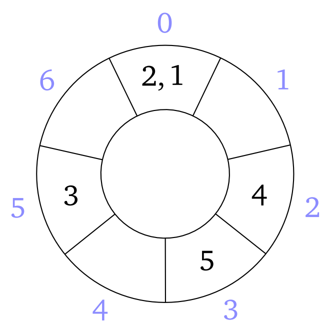


Figure 2.2: Example of circular placement.

The numbers in a box are arranged one after the other in the clockwise direction. A circular placement is called 'ordered' if the numbers in each box are arranged in ascending order.

We will work with circular placements without labels on the boxes. We think of them as a circular arrangement in $\overline{\mathbb{Z}_q^n}$, that is, $\mathbb{Z}_q^n / H$ where H is the subgroup generated by $(1, \ldots, 1) \in \mathbb{Z}_q^n$. So we can choose some box to be 0 and label the others cyclically in the clockwise direction to get an element of $\mathbb{Z}_q^n$. Hence the ordered circular placements are in bijection with the quotient group $\mathbb{Z}_q^n / H = \overline{\mathbb{Z}_q^n}$.

Remark 2.2. The properties of $x_i - x_j$ do not depend on the choice of the 0 box in a circular placement in $\overline{\mathbb{Z}_q^n}$.

Given any circular placement in $\overline{\mathbb{Z}_q^n}$, we can obtain a circular placement in $\overline{\mathbb{Z}_{q+n}^n}$ by adding a line immediately after each number in the placement. The resulting placement in $\overline{\mathbb{Z}_{q+n}^n}$ will have at most one number in each box.

From a circular placement in $\overline{\mathbb{Z}_{q+n}^n}$ with at most one number in each box, we get a circular placement in $\overline{\mathbb{Z}_q^n}$ by removing the line immediately following each number. These operations give us a bijection between circular placements in $\overline{\mathbb{Z}_q^n}$ and circular placements in $\overline{\mathbb{Z}_{q+n}^n}$ with at most one number in each box.

Example 2.3. Figure 2.3 shows as example of this bijection.

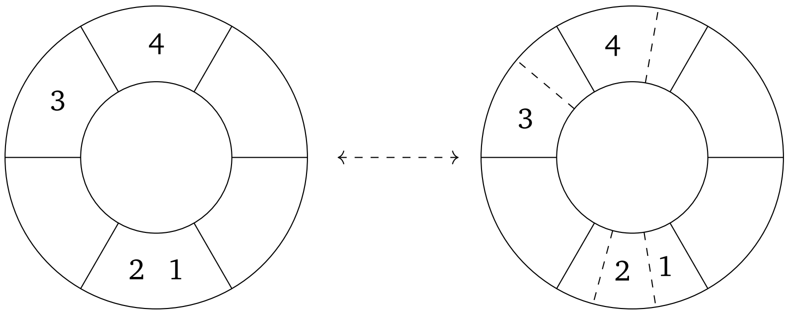


Figure 2.3: Example of the bijection between circular placements

We record some properties of this bijection. Let P be a circular placement in $\overline{\mathbb{Z}_q^n}$ and Q be its corresponding placement in $\overline{\mathbb{Z}_{q+n}^n}$. Let $\tilde{P} = (x_1, \ldots, x_n)$ and $\tilde{Q} = (x'_1, \ldots, x'_n)$ be one of the corresponding tuples in $\mathbb{Z}_q^n$ and $\mathbb{Z}_{q+n}^n$ for P and Q respectively. Then,

1. $x'_i - x'_j \neq 0 \quad \forall\, 1 \leq i < j \leq n.$
2. $x'_i - x'_j \neq 0, 1 \quad \forall\, 1 \leq i < j \leq n \Leftrightarrow P$ is ordered.
3. $x'_i - x'_j \neq 0, 1, 2 \quad \forall\, 1 \leq i < j \leq n \Leftrightarrow P$ is ordered and $x_i - x_j \neq 1 \quad \forall\, 1 \leq i < j \leq n.$

Hence, the cardinality of $\{\overline{(x_1, \ldots, x_n)} \in \overline{\mathbb{Z}_q^n} \mid x_i - x_j \neq 1 \quad \forall\, 1 \leq i < j \leq n\}$ is the same as that of $\{\overline{(x_1, \ldots, x_n)} \in \overline{\mathbb{Z}_{q+n}^n} \mid x_i - x_j \neq 0, 1, 2 \quad \forall\, 1 \leq i < j \leq n\}$.

Remark 2.3. A circular placement with at most one number in each block is ordered.

In fact, we can extend this argument, using the same bijection as above, to prove that for any integer $m \geq 1$:

> The cardinality of $\{\overline{(x_1, \ldots, x_n)} \in \overline{\mathbb{Z}_q^n} \mid x_i - x_j \neq 1, \ldots, m \quad \forall\, 1 \leq i < j \leq n\}$ is the same as that of $\{\overline{(x_1, \ldots, x_n)} \in \overline{\mathbb{Z}_{q+n}^n} \mid x_i - x_j \neq 0, 1, \ldots, m+1 \quad \forall\, 1 \leq i < j \leq n\}$.

We do this by extending the list of properties of the bijection so that the k^{th} property of the list for $k > 2$ is:

k. $x'_i - x'_j \neq 0, \ldots, k-1 \quad \forall\, 1 \leq i < j \leq n \Leftrightarrow P$ is ordered and $x_i - x_j \neq 1, \ldots, k-2 \quad \forall\, 1 \leq i < j \leq n.$

Note that we can derive the $(k+1)^{th}$ property from the k^{th} property as follows:

The backward implication of property $(k+1)$ follows from the fact that the number of boxes between two numbers increases by at least one when we add a line after each number.

For the forward implication, since we are assuming property k is true, we only have to prove that if the following hold then $x'_i - x'_j = k$ for some $1 \leq i < j \leq n$:

- P is ordered.
- $x_i - x_j \neq 1, \ldots, k-2 \quad \forall\, 1 \leq i < j \leq n$.
- $x_i - x_j = k-1$ for some $1 \leq i < j \leq n$.

Suppose $1 \leq i_1 < j_1 \leq n$ and $x_{i_1} - x_{j_1} = k-1$. Let i_0 be the first number in the box containing i_1 and j_0 be the last number in the box containing j_1. Since P is ordered, $i_0 \leq i_1$ and $j_0 \geq j_1$. Since $x_i - x_j \neq 1, \ldots, k-2 \quad \forall\, 1 \leq i < j \leq n$, if there is a number in the $k-2$ boxes between j_0 and i_0, it must be $\geq j_0$ and $\leq i_0$. Since $i_0 < j_0$, there are $k-2$ empty boxes between j_0 and i_0. When we draw a line after all numbers, there will be $k-1$ boxes between j_0 and i_0. So, $x'_{i_0} - x'_{j_0} = k$.

Recall that the arrangement in $\mathbb{R}^n$ with hyperplanes:

$$\{x_i - x_j = 0, \ldots, m \mid 1 \leq i < j \leq n\}$$

for any integer $m \geq 1$, is denoted by $\mathcal{A}_{[0,m]}(n)$. We will denote it by $\mathcal{A}_n^m$ in this chapter.

Also, denote by $\mathscr{L}_n^m$ the extended Linial arrangement in $\mathbb{R}^n$:

$$\{x_i - x_j = 1, \ldots, m \mid 1 \leq i < j \leq n\}$$

for any integer $m \geq 1$.

For any subset T of integers,

$$\begin{aligned}&\#\{(x_1, \ldots, x_n) \in \mathbb{Z}_q^n \mid x_i - x_j \neq t \quad \forall\, t \in T,\ 1 \leq i < j \leq n\}\\ &= q \times \#\{\overline{(x_1, \ldots, x_n)} \in \overline{\mathbb{Z}_q^n} \mid x_i - x_j \neq t \quad \forall\, t \in T,\ 1 \leq i < j \leq n\}.\end{aligned}$$

From the above observations and the finite field method, we get:

$$\frac{\chi(\mathscr{L}_n^m, q)}{q} = \frac{\chi(\mathcal{A}_n^{m+1}, q+n)}{q+n}.$$

So if we can obtain the characteristic polynomial of the arrangement $\mathcal{A}_n^m$ for all $m \geq 2$, we will solve our problem for extended Linial arrangements. We will do this in Section 2.5.

2.4 The extended Shi arrangement

Denote by $\mathcal{S}_n^m$ the extended Shi arrangement in $\mathbb{R}^n$:

$$\{x_i - x_j = -m, \ldots, m+1 \mid 1 \leq i < j \leq n\}$$

for any integer $m \geq 1$. Notice that the $x_i - x_j = 0$ type hyperplanes are included in $\mathcal{S}_n^m$ for all $m \geq 1$.

We apply the finite field method as usual. Just as with the extended Linial arrangements, we look at circular placements in $\overline{\mathbb{Z}_q^n}$. The computation of the characteristic polynomial in this section follows the method in [3].

We have to count the placements P such that if $\tilde{P} = (x_1, \ldots, x_n)$ is a corresponding tuple,

$$x_i - x_j \neq -m, \ldots, m+1 \quad \forall\, 1 \leq i < j \leq n.$$

That is,

$$\begin{array}{c} x_i \neq x_j + k \quad \forall\, k \in \{1, \ldots, m+1\} \\ x_j \neq x_i + k \quad \forall\, k \in \{0, \ldots, m\} \end{array} \qquad \forall\, 1 \leq i < j \leq n.$$

These are the circular placements in $\overline{\mathbb{Z}_q^n}$ such that there is at most one number in each box and if l_1 and l_2 are consecutive numbers in the placement (read clockwise)

- If $l_1 < l_2$, there should be at least m boxes between them.
- If $l_2 < l_1$, there should be at least $m+1$ boxes between them.

By removing m boxes after each number, we get a bijection with circular placements in $\overline{\mathbb{Z}_{q-mn}^n}$ where no two numbers are in the same box and if l_1 and l_2 are consecutive in the placement,

- If $l_1 < l_2$, there should be at least 0 boxes between them. (That is, there is no restriction on number of boxes between them).
- If $l_2 < l_1$, there should be at least 1 box between them.

But these are the type of circular placements counted in the usual Shi arrangement. Hence, using the same observations as before, we get:

$$\frac{\chi(\mathcal{S}_n^m, q)}{q} = \frac{\chi(\mathcal{S}_n, q - mn)}{q - mn}.$$

So we get, $\chi(\mathcal{S}_n^m, q) = q(q - mn - n)^{n-1} = q(q - (m+1)n)^{n-1}$.

2.5 The extended Linial arrangement

By the observations made in Section 2.3, we just need to calculate the characteristic polynomial of the arrangement $\mathcal{A}_n^m$. Though the value of the count is more complicated than in the case of the Shi arrangement ($m = 1$), the idea is almost the same. The computation of the characteristic polynomial of $\mathcal{A}_n^m$ in this section follows the method in [3].

We use the same "ball-number" representations as before. Just as before, we count the placements with $x_n = 0$ and multiply the result by q.

The conditions we have are:

$$x_i - x_j \neq 0, \ldots, m \quad \forall\, 1 \leq i < j \leq n.$$

So we have to count the placements (with n at the beginning) such that if i is the first number after j in the placement and $i < j$, there must be at least m balls between them. Call such placements valid.

To help in counting such placements, we define the following:

Given a placement and two consecutive numbers in it, we define the number of m-blocks between them as follows: If there are p balls between them, then $p = ms + r$ for unique integers s, r such that $0 \le r < m$. We say there are s m-blocks between the numbers ($s = \lfloor \frac{p}{m} \rfloor$).

So the valid placements must have at least one m-block between consecutive numbers j and i if $i < j$. (2.5)

What we will do is count the number of valid placements with a fixed total number of m-blocks.

Let j be the number of m-blocks. We perform the count as follows: We start with a diagram of n followed by mj balls (representing the j m-blocks) as in Figure 2.4.

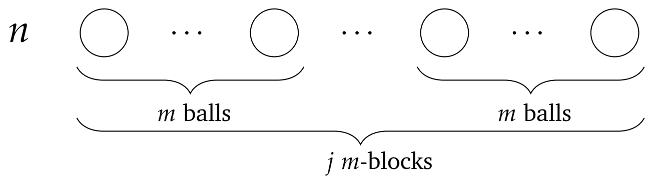


Figure 2.4: Taking number of m-blocks as j and $x_n = 0$.

Then we see how many ways we can put in the numbers $1, \dots, n-1$ in the spaces between the m-blocks or after the last m-block. By 2.5, we cannot place any number after n and before the first m-block. Also, because of 2.5, the numbers put in any space (between two m-blocks or after the last one) should be in ascending order. After placing the numbers $1, \dots, n-1$, we have to put in the remaining $q - mj - n$ balls. We have assumed that n is in the first position, so there will be no balls before it. Only the order in which the numbers appear and the number of balls after each number matters (remember that the balls are identical). So each of the remaining $q - mj - n$ balls can be put in any of n spaces after the numbers $1, \dots, n$ and at most $m - 1$ can be put in each space (so that the number of m-blocks remains j).

So, the total number of valid placements with j m-blocks is

$$j^{n-1} \times \#\{(b_1, \dots, b_n) \mid 0 \le b_i \le m-1 \; \forall \, i \in [n], \; \sum_{i=1}^{n} b_i = q - mj - n\}.$$

Here the first term is the number of ways to place the numbers $1, \dots, n-1$ and the second is the number of ways to place the remaining $q - mj - n$ balls (regard b_i as the number of balls out of the remaining $q - mj - n$ being placed after i).

We can see that this is the same as

$$[y^{q-n}]\left((1 + y + \cdots + y^{m-1})^n \times j^{n-1} y^{mj} \right).$$

where $[y^k]F(y)$ is the coefficient of y^k in $F(y)$.

So the number of valid placements is

$$[y^{q-n}]\left((1 + y + \cdots + y^{m-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{mj} \right).$$

where the term in brackets is treated as a formal power series (refer [20]). Call this number $VP(q)$.

So, for large q,

$$\chi(\mathcal{A}_n^m, q) = qVP(q).$$

Remember that we were counting the placements with $x_n = 0$, so the factor of q has to be multiplied.

In fact, it can be shown that for large q ($q > mn$), $VP(q)$ is a polynomial in q. This follows because for $p > k$,

$$[y^p]\left(\left(\sum_{i=0}^{k} c_i y^i\right)\left(\sum_{j=0}^{\infty}(mj)^{n-1}y^{mj}\right)\right) = \sum_{\substack{i \equiv p \pmod a \\ i \in \{0,\ldots,m\}}} c_i (p-i)^{n-1}.$$

Hence we have obtained the characteristic polynomial of $\mathcal{A}_n^m$ and hence, by the arguments of Section 2.3, of $\mathscr{L}_n^m$ for any $m \geq 1$. After some algebraic manipulations and using Zaslavsky's theorem, we can obtain the number of regions and bounded regions of the arrangements mentioned in this chapter. This is given in Table 2.1. The term b_k in the expression for the m-Linial region numbers is defined for any integer k as

$$b_k = [y^k](1 + y + y^2 \cdots + y^m)^n.$$

Name	$\mathcal{A}$	$r(\mathcal{A})$	$b(\mathcal{A})$
Braid	$\mathcal{A}_{\{0\}}(n)$	$n!$	0
Shi	$\mathcal{A}_{\{0,1\}}(n)$	$(n+1)^{n-1}$	$(n-1)^{n-1}$
Linial	$\mathcal{A}_{\{1\}}(n)$	$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}$	$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k-1)^{n-1}$
m-Shi	$\mathcal{A}_{[-m+1,m]}(n)$	$(mn+1)^{n-1}$	$(mn-1)^{n-1}$
m-Linial	$\mathcal{A}_{[1,m]}(n)$	$\frac{1}{(m+1)^n}\sum_{k=0}^{n} b_k(k+1)^{n-1}$	$\frac{1}{(m+1)^n}\sum_{k=0}^{n} b_k(k-1)^{n-1}$

Table 2.1: Number of regions for arrangements considered in this chapter.

Chapter 3

Classical bijections

In this chapter we describe specific bijections for the regions of the braid, Catalan and Shi arrangements. The one for the Catalan regions is with labeled balanced bracket systems. This bijection can be found in [5]. The bijection for the Shi regions, due to Athanasiadis and Linusson [4], is with parking functions. Both these bijections can be extended to ones for the extended Catalan and Shi arrangements. In each case, the goal is to obtain a bijection between the regions and some set that is easier to count or whose count is already known.

3.1 The braid arrangement

Recall that the braid arrangement in $\mathbb{R}^n$, denoted by $\mathcal{A}_{\{0\}}(n)$, is given by:

$$\{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}.$$

Any region of the braid arrangement is of the form $x_{\sigma(1)} < \cdots < x_{\sigma(n)}$ for some permutation σ of $[n]$. This is because validly choosing $x_i < x_j$ or $x_i > x_j$ for all $1 \leq i < j \leq n$ gives a total order on the coordinates. So we see that there is a bijection between the regions of $\mathcal{A}_{\{0\}}(n)$ and the permutations of $[n]$.

3.2 The Catalan arrangement

Recall that the Catalan arrangement in $\mathbb{R}^n$ is given by:

$$\{x_i - x_j = -1, 0, 1 \mid 1 \leq i < j \leq n\}.$$

We will denote it by $\mathscr{C}_n$ instead of $\mathcal{A}_{\{-1,0,1\}}(n)$.

Due to the symmetry of the set $\{-1, 0, 1\}$, we can write $\mathscr{C}_n$ as:

$$\{x_i = x_j,\, x_i = x_j + 1 \mid i, j \in [n],\, i \neq j\}.$$

Looking at the Catalan arrangement in this way, we see that specifying a region of $\mathscr{C}_n$ is the same as specifying a 'valid total order' on $x_1, \ldots, x_n, x_1 + 1, \ldots, x_n + 1$.

Definition 3.1 (Valid total order)**.** A total order $l_1(\boldsymbol{x}) < \cdots < l_k(\boldsymbol{x})$ where each l_i is a linear form on $\mathbb{R}^n$ is called valid if there is a point $\boldsymbol{x_0} \in \mathbb{R}^n$ such that $l_1(\boldsymbol{x_0}) < \cdots < l_k(\boldsymbol{x_0})$.

Example 3.1. $x_1 < x_2 < x_3 < x_1 + 1 < x_2 + 1 < x_3 + 1$ is a valid order. Since (0,0.1,0.2) satisfies this order. However, $x_1 < x_2 < x_2 + 1 < x_1 + 1 < x_3 < x_3 + 1$ is not a valid order.

We represent such orders by a word with the $2n$ letters in

$$A^{(1)}(n) = \{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\}.$$

Here, $\alpha_i^{(s)}$ represents $x_i + s$. The letters $\alpha_i^{(0)}$ are called α-letters and $\alpha_i^{(1)}$ are called β-letters (this terminology will become clear later).

Example 3.2. The valid order $x_3 < x_2 < x_3 + 1 < x_2 + 1 < x_1 < x_1 + 1$ would be represented as

$$\alpha_3^{(0)}\alpha_2^{(0)}\alpha_3^{(1)}\alpha_2^{(1)}\alpha_1^{(0)}\alpha_1^{(1)}.$$

Definition 3.2 (1-sketch of size n)**.** A word in the letters $A^{(1)}(n)$ which corresponds to a valid total order on the symbols $x_1, \ldots, x_n, x_1 + 1, \ldots, x_n + 1$ is called a 1-sketch of size n. Hence, 1-sketches of size n correspond to regions of $\mathscr{C}_n$.

It is clear that not all words in $A^{(1)}(n)$ correspond to regions. For example: $\alpha_1^{(0)}\alpha_2^{(0)}\alpha_2^{(1)}\alpha_1^{(1)}$ is not a 1-sketch since $x_1 < x_2 < x_2 + 1 < x_1 + 1$ is not a valid order.

Proposition 3.1. *A word in $A^{(1)}(n)$ is a 1-sketch if and only if*

1. *Each letter in $A^{(1)}(n)$ appears exactly once.*
2. *For any $i, j \in [n]$, $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)} \Rightarrow \alpha_i^{(1)}$ appears before $\alpha_j^{(1)}$.*
3. *For any $i \in [n]$, $\alpha_i^{(0)}$ appears before $\alpha_i^{(1)}$.*

To prove this we must show that there is a point in $\mathbb{R}^n$ satisfying the inequalities given by such a word. We will prove this in greater generality in Proposition 3.3.

From property 2, it is clear that if the subscripts appear in order $\sigma(1), \ldots, \sigma(n)$ for the (0)-type letters in a 1-sketch, then they appear in the same order for the (1)-type letters. Hence, a 1-sketch is specified by a permutation σ and a word with n α's and n β's. (Here the α's represent the (0)-type letters and β's the (1)-type letters). By property 3, the number of α's in any prefix of the word is greater than or equal to the number of β's. Hence we get the following:

Proposition 3.2. *A 1-sketch of size n is completely specified by:*

1. *A permutation on $[n]$.*
2. *A word with n α's and n β's such that in any prefix the number of α's is greater than or equal to the number of β's.*

Example 3.3. $\alpha_2^{(0)}\alpha_1^{(0)}\alpha_2^{(1)}\alpha_1^{(1)}\alpha_3^{(0)}\alpha_3^{(1)}$ would correspond to the permutation 213 and the word $\alpha\alpha\beta\beta\alpha\beta$, which we can think of as the tuple $(213, \alpha\alpha\beta\beta\alpha\beta)$. For convenience this can also be written as $\alpha_2\alpha_1\beta\beta\alpha_3\beta$.

Since a sequence of n left and n right brackets is balanced if and only if the number of left brackets in any prefix is greater than or equal to the number of right brackets, we get a bijection from 1-sketches to tuples of the form (permutation, balanced bracket system). And since the number of balanced bracket systems of length $2n$ is the n^{th} Catalan number (refer [22]), we get:

$$r(\mathscr{C}_n) = n! \times \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!}.$$

Using similar ideas we can get that the regions of the m-Catalan arrangement in $\mathbb{R}^n$ are in bijection with words having letters in $A^{(m)}(n) = \{\alpha_i^{(s)} \mid i \in [n],\ s \in [0,m]\}$ such that:

1. Each letter in $A^{(m)}(n)$ appears exactly once.
2. For any $i, j \in [n]$ and $s, t \in [m]$,

$$\alpha_i^{(s-1)} \text{ appears before } \alpha_j^{(t-1)} \Rightarrow \alpha_i^{(s)} \text{ appears before } \alpha_j^{(t)}.$$

3. For any $i \in [n]$ and $s \in [m]$, $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

Such words are called m-sketches. Our aim now is to show that there is a point in $\mathbb{R}^n$ satisfying the order given by an m-sketch.

Remark 3.1. The method given in [5] for constructing such a point does not seem to work in general. We first describe their method below and then exhibit the problem in the method.

Let $w = w_1 \dots w_{(m+1)n}$ be an m-sketch. Then construct $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ as follows. Let $z_0 = 0$ (or pick z_0 arbitrarily). Then define z_p for $p = 1, \dots, (m+1)n$ in order as follows: If $w_p = \alpha_i^{(0)}$ then set $z_p = z_{p-1} + \frac{1}{n+1}$ and $x_i = z_p$, and if $w_p = \alpha_i^{(s)}$ with $s \neq 0$ then set $z_p = x_i + s$. Then $\boldsymbol{x}$ satisfies the inequalities given by w.

The following example shows that this method does not always work:

Consider the 1-sketch $w = \alpha_1^{(0)}\alpha_2^{(0)}\alpha_1^{(1)}\alpha_3^{(0)}\alpha_2^{(1)}\alpha_3^{(1)}$. By the above procedure we would get:

- $w_1 = \alpha_1^{(0)}$ so $z_1 = \frac{1}{4}$, $x_1 = \frac{1}{4}$.
- $w_2 = \alpha_2^{(0)}$ so $z_2 = \frac{2}{4}$, $x_2 = \frac{2}{4}$.
- $w_3 = \alpha_1^{(1)}$ so $z_3 = 1 + \frac{1}{4}$.
- $w_4 = \alpha_3^{(0)}$ so $z_4 = 1 + \frac{2}{4}$, $x_3 = 1 + \frac{2}{4}$.
- $w_7 = \alpha_2^{(1)}$ so $z_5 = 1 + \frac{2}{4}$.
- $w_8 = \alpha_3^{(1)}$ so $z_6 = 2 + \frac{2}{4}$.

Hence $\boldsymbol{x} = (\frac{1}{4}, \frac{2}{4}, 1 + \frac{2}{4})$. But this $\boldsymbol{x}$ does not satisfy $x_3 < x_2 + 1$ (which is the inequality given by $\alpha_3^{(0)}$ being before $\alpha_2^{(1)}$). In fact, the same example will show that replacing $\frac{1}{n+1}$ with some other positive constant will also not work.

We will now state an alternate proof for the existence of such a point. The idea is to choose the coordinates of the point one by one in the order on the coordinates specified by the m-sketch. Before giving a general proof, we first look at an example.

Example 3.4. Consider the 2-sketch $\alpha_2^{(0)}\alpha_1^{(0)}\alpha_2^{(1)}\alpha_1^{(1)}\alpha_2^{(2)}\alpha_3^{(0)}\alpha_1^{(2)}\alpha_3^{(1)}\alpha_3^{(2)}$. We will choose a point $(a_1, a_2, a_3) \in \mathbb{R}^3$ satisfying the required inequalities by choosing a_2, then a_1 and finally a_3 (see figures below). a_2 is chosen arbitrarily. After doing so, $a_2, a_2 + 1$ and $a_2 + 2$ are marked off on the number line. Then a_1 is chosen in the correct position with respect to $a_2, a_2 + 1$ and $a_2 + 2$ (between a_2 and $a_2 + 1$) and again the corresponding numbers and marked off. Finally a_3 is chosen in the correct relative position to the marked off numbers (between $a_2 + 2$ and $a_1 + 2$). The choices $a_2 = 0, a_1 = 0.5$ and $a_3 = 2.25$ done in this way is shown in Figures 3.1, 3.2, 3.3.

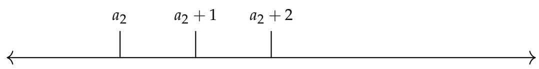


Figure 3.1: Choosing a_2 arbitrarily and marking off $a_2, a_2 + 1, a_2 + 2$.

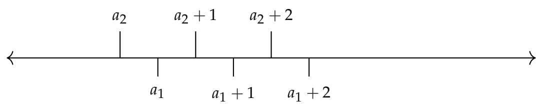


Figure 3.2: Choosing a_1 in correct position and marking off $a_1, a_1 + 1, a_1 + 2$.

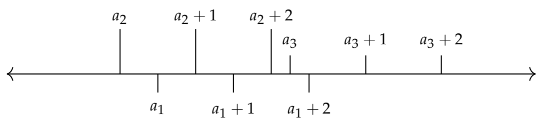


Figure 3.3: Choosing a_3 in correct position and marking off $a_3, a_3 + 1, a_3 + 2$.

We will now prove that this method works in general.

Proposition 3.3. *There is a point in $\mathbb{R}^n$ that satisfies the inequalities specified by an m-sketch.*

Proof. Let w be an m-sketch. Without loss of generality, we can assume that the order in which the α-letters appear is $\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_n^{(0)}$. We will construct a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ satisfying the inequalities given by w.

Denote by $w|_k$, for any $k \in [n]$, the restriction of w to k, that is, the word obtained by removing the letters $\alpha_i(s)$ for any $i > k$. Hence this $w|_k$ gives an order on $x_1, \ldots, x_k, x_1 + 1, \ldots, x_k + 1, \ldots, x_1 + m, \ldots, x_k + m$. The idea is to choose $a_1, \ldots, a_n$ in order so that $(a_1, \ldots, a_k)$ satisfy the inequalities given by $w|_k$ for all $k \in [n]$. Also, when a_i is chosen, $a_i, a_i + 1, \ldots, a_i + m$ are marked off on the number line.

Choose a_1 arbitrarily. (a_1) satisfies $w|_1$. Suppose $(a_1, \ldots, a_k)$ have been chosen to satisfy $w|_k$. Looking at the number line, choose a_{k+1} in the correct relative position to $a_1, \ldots, a_1 + m, \ldots, a_k, \ldots, a_k + m$. After doing so, mark off $a_{k+1} + 1, \ldots, a_{k+1} + m$ as well. We claim that $(a_1, \ldots, a_{k+1})$ satisfies $w|_{k+1}$.

To prove this, we just need to check that $a_{k+1},\ a_{k+1} + 1, \ldots,\ a_{k+1} + m$ are in the correct position relative to $a_1, \ldots,\ a_1 + m, \ldots,\ a_k, \ldots,\ a_k + m$. This is because $a_{k+1},\ a_{k+1} + 1, \ldots,\ a_{k+1} + m$ are already in the correct position with respect to each other. By choice of a_{k+1}, it is in the correct relative position.

Suppose $a_{k+1}, \ldots,\ a_{k+1} + (s-1)$ are in the correct relative position (with respect to $a_1, \ldots,\ a_1 + m, \ldots,\ a_k, \ldots,\ a_k + m$) for some $s \geq 1$. If $a_{k+1} + s$ is not in the correct relative position, then one of the following must hold:

1. $a_{k+1} + s$ is before $a_i + t$ but $\alpha_{k+1}^{(s)}$ is after $\alpha_i^{(t)}$ for some $i \in [k]$ and $t \in [m]$.

2. $a_{k+1} + s$ is after $a_i + t$ but $\alpha_{k+1}^{(s)}$ is before $\alpha_i^{(t)}$ for some $i \in [k]$ and $t \in [m]$.

We have $t \in [m]$ because, for any $i \in [k]$, a_{k+1} is after a_i and hence we cannot have $a_{k+1} + s$ before some a_i and since $\alpha_{k+1}^{(0)}$ is after $\alpha_i^{(0)}$, so is $\alpha_{k+1}^{(s)}$.

If 1 holds, we have

$$a_{k+1} + s < a_i + t \Rightarrow a_{k+1} + (s-1) < a_i + (t-1)$$

and by a property of m-sketches,

$$\alpha_{k+1}^{(s)} \text{ after } \alpha_i^{(t)} \Rightarrow \alpha_{k+1}^{(s-1)} \text{ after } \alpha_i^{(t-1)}.$$

But this would contradict the fact that $a_{k+1} + (s-1)$ is in the correct relative position.

A similar argument works in the case when 2 holds. □

3.3 The Shi arrangement

Recall that the Shi arrangement in $\mathbb{R}^n$ is given by:

$$\{x_i - x_j = 0, 1 \mid 1 \leq i < j \leq n\}.$$

We will denote it by $\mathcal{S}_n$ instead of $\mathcal{A}_{\{0,1\}}(n)$.

Using the finite field method, we already know that $\mathcal{S}_n$ has $(n+1)^{n-1}$ regions. This is also the number of parking functions of length n. In this section, we will define and count parking functions and then obtain a bijection between these objects.

3.3.1 Parking functions

Definition 3.3 (Parking function of length n)**.** A sequence $(a_1, \ldots, a_n)$ of numbers in $[n]$ such that if $(b_1, \ldots, b_n)$ is the sequence in ascending order, we get $b_i \leq i$ for all $i \in [n]$. That is, at least i numbers in $(a_1, \ldots, a_n)$ are less than or equal to i for all $i \in [n]$.

Example 3.5.

- $(2,6,3,1,4,1)$ is a parking sequence of length 6 since in ascending order it is $(1,1,2,3,4,6)$.
- $(6,1,3,5,1,5,6)$ is not a parking function since it has only 3 terms less than or equal to 4.

These functions are called parking functions because there is an equivalent "parking definition" which we state now.

Consider the following scenario:

There are n cars that attempt to park in n spots in order. That is, the cars, which we call Car 1, ..., Car n, attempt to park in Spot 1, ..., Spot n with Car 1 attempting first, then Car 2, and so on.

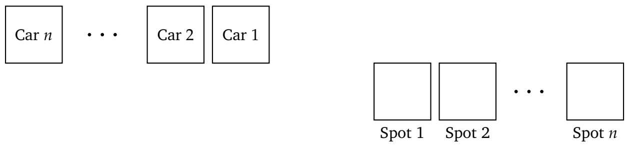


Figure 3.4: n cars and n spots.

Each car has a preferred spot. Let a_i be the preferred spot of Car i . Each car (in order) drives up to its preferred spot and parks there if it is empty. If it is occupied, it parks in the first empty spot after it. If no such spot exists, it drives off.

A sequence $(a_1,\ldots,a_n)$ for which all cars end up parked is called a parking function of length n. The sequence $(a_1,\ldots,a_n)$ in this parking situation is called the preference sequence.

Example 3.6. The parking for preference sequence (2,1,1,3) would go as shown in Figure 3.5.

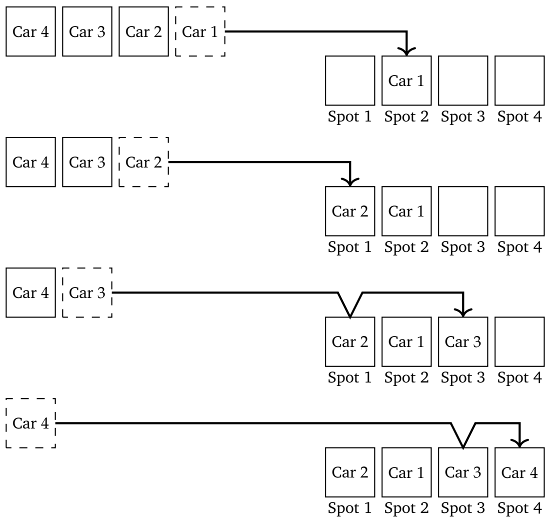


Figure 3.5: Parking for the preference sequence $(2,1,1,3)$.

Notice that in this case (2,1,1,3) is a parking function by both definitions.

We will now prove that both these definitions are equivalent. That is, if $(a_1, \ldots, a_n)$ is a preference sequence and $(b_1, \ldots, b_n)$ is the sequence is ascending order,

$$\text{All cars can park} \Leftrightarrow b_i \leq i \text{ for all } i \in [n].$$

First note that if Car k could not park, then all spots $\geq a_k$ have to be filled. We will prove that:

$$\text{At least one car can't park} \Leftrightarrow b_i > i \text{ for some } i \in [n].$$

Now, suppose some car can't park and Spot i is last vacant spot after parking is done (since there are equal number of spots as cars, at least one will be empty). This will mean that the cars in spot $i+1, \ldots, n$ and the car(s) that left prefer spots strictly greater than i. Since otherwise Spot i would be filled. But this would mean at least $n-i+1$ cars prefer spots strictly greater than i. Hence, $b_i, \ldots, b_n > i$ since these are the last $n-i+1$ terms in the ascending sequence. Hence we get $b_i > i$.

Conversely, suppose there is some $i \in [n]$ such that $b_i > i$. For such an i, we will have $b_i, \ldots, b_n > i$. Hence the $n-i+1$ cars corresponding to these terms all prefer spots strictly greater than i. But there are only $n-i$ such spots. Hence at least one of these cars won't be able to park.

The reason we have mentioned this equivalent definition is that it gives a nice proof for the fact that there are $(n+1)^{n-1}$ parking functions of length n.

To count the parking functions, we will consider a slightly different situation from that described in the parking definition.

We consider the spots on a circle instead of on a line. Also, there is an extra spot $n+1$. The number of cars is still n but we allow cars to prefer Spot $n+1$. The method of parking is just as before (first Car 1 attempts to park, then Car 2, and so on) but in this situation, the cars move in a clockwise direction to find a spot (instead of linearly as before).

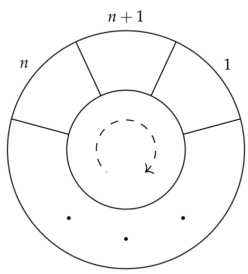


Figure 3.6: $n+1$ spots placed along a circle.

Hence there is no possibility of a car not being able to park. So all cars park and there will be exactly one empty spot after the parking is done. In this situation, the preference sequences are elements of $[n+1]^n$ which we consider as $\mathbb{Z}_{n+1}^n$.

We claim that the preference sequence $(a_1, \ldots, a_n)$ in this situation is a parking function of length n if and only if the spot that is empty after parking is $n+1$.

If $(a_1, \ldots, a_n)$ is a not parking function, this can happen in two ways. Either some $a_i = n+1$ and hence Car i or some car before it will park in Spot $n+1$. Or all $a_i \in [n]$ but in the linear order of the spots, all cars won't be able to park. In this situation, the first car that would have left the parking lot in the linear case will fill Spot $n+1$. Hence in any case, if $(a_1, \ldots, a_n)$ is not a parking function, Spot $n+1$ will be filled.

If $(a_1, \ldots, a_n)$ is a parking function, the Spot $n+1$ will be empty since the cars never use the circular structure of the parking lot. They park just as they would have in the linear case.

Now our task is to count the preference sequences for which Spot $n+1$ is left empty.

Due to the circular structure of the parking lot, if Spot i is left empty after parking for the preference sequence $(a_1, \ldots, a_n)$, then for any $k \in [n]$, Spot $i+k$ is left empty after parking for the preference sequence $(a_1 + k, \ldots, a_n + k)$. Here all the additions are modulo $n+1$.

So we get that for any preference sequence $(a_1, \ldots, a_n)$, exactly one of the elements of $\{(a_1 + k, \ldots, a_n + k) : k = 0, \ldots, n\}$ is a parking function (since exactly one will have Spot $n+1$ empty after parking). But sets of this form partition $\mathbb{Z}_{n+1}^n$. Since each has exactly one parking function, and there are $(n+1)^{n-1}$ such sets, we get our desired count.

3.3.2 The bijection

We will first construct a method of representing regions of the Shi arrangement.

Recall that $\mathcal{S}_n = \{x_i - x_j = 0, 1 \mid 1 \leq i < j \leq n\}$. And a region is given by validly choosing :

$$x_i - x_j > 0 \text{ or } x_i - x_j < 0 \text{ for each } 1 \leq i < j \leq n$$
$$\text{and}$$
$$x_i - x_j > 1 \text{ or } x_i - x_j < 1 \text{ for each } 1 \leq i < j \leq n.$$

We describe a method of representing the choice of these inequalities. We do so using arc diagrams, that is, diagrams with numbers in some order and arcs between some numbers.

The $x_i - x_j = 0$ type inequalities give us a total order on the coordinates (since the region is nonempty). If $x_{\sigma(1)} > \cdots > x_{\sigma(n)}$ we write this as:

$$\sigma(1) \;\; \sigma(2) \;\; \ldots \;\; \sigma(n)$$

Now we represent the $x_i - x_j = 1$ $(i < j)$ type inequalities by drawing arcs. We only need to specify these inequalities for $x_i - x_j$ where $i < j$ and i is before j in the diagram. If $i < j$ and j is before i in the diagram we know $x_i - x_j < 0$ and hence $x_i - x_j < 1$. But if i is before j, we can either have $0 < x_i - x_j < 1$ or $x_i - x_j > 1$. For such i, j draw an arc from i to j if $x_i - x_j > 1$. After drawing all these arcs, we erase any arc that contains another. This is because these inequalities are implied by the inner arc. If there is an arc from i to j (hence $i < j$) and i' is before i and j' is after j and $i' < j'$, then there must be an arc from i' to j' ($x_{i'} \leq x_i < x_j \leq x_{j'}$ and $x_i - x_j > 1 \Rightarrow x_{i'} - x_{j'} > 1$).

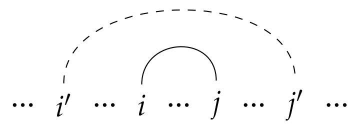


Figure 3.7: Redundant arcs.

From the way the arc diagram was constructed, we can get back the inequalities that define the region. Using the order in which the number appear, we can get back the $x_i - x_j = 0$ type inequalities. For any $i < j$, if j appear before i in the diagram, we get $x_i - x_j < 1$. If i appears before j, then if we imagine an arc from i to j and it contains some arc of the diagram, we have $x_i - x_j > 1$ and if it does not, we get $x_i - x_j < 1$.

Example 3.7. Consider the region in $\mathcal{S}_4$ given by: $x_2 > x_1 > x_3 > x_4$, $x_1 - x_2 < 1$, $x_1 - x_3 < 1$, $x_1 - x_4 > 1$, $x_2 - x_3 > 1$, $x_2 - x_4 > 1$, $x_3 - x_4 < 1$. We obtain the associated arc diagram via the steps in Figure 3.8.

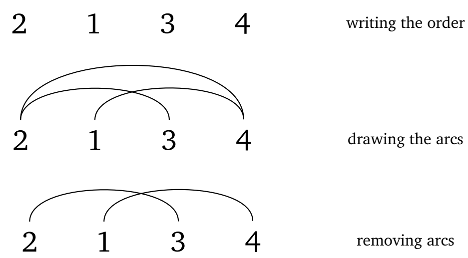


Figure 3.8: Example of constructing the arc diagram associated to a region.

We now have to obtain a bijection between arc diagrams that correspond to regions of $\mathcal{S}_n$ and parking functions of length n. So we first have to characterize the arc diagrams that correspond to regions.

We first note some properties that such arc diagrams will have to satisfy and it turns out that these properties are sufficient for an arc diagram to correspond to a region:

1. The numbers $1, \ldots, n$ appear in some order.

2. Arcs are only from smaller to larger numbers.

3. Arcs do not contain other arcs.

We call arc diagrams that satisfy these properties valid arc diagrams of length n. To show that all valid arc diagrams correspond to regions, we have to show that there is a point in $\mathbb{R}^n$ that satisfies the inequalities specified by the arc diagram. The idea is to exhibit a Catalan region which satisfies all the inequalities given by such an arc diagram. We will just state a method of constructing such a Catalan region and leave the verification of the details to the reader.

First note that even if a word in $A^{(1)}(n)$ does not contain all the β-letters but satisfies property 2 and 3 of 1-sketches (see Proposition 3.1), it still specifies a nonempty subset of $\mathbb{R}^n$

(one way to prove this is to add the missing β-letters in such a way that a 1-sketch is obtained). Our goal is to construct such a partial 1-sketch which satisfies the inequalities of a given valid arc diagram.

Let V be a valid arc diagram. Let $a_n\ a_{n-1}\ \ldots\ a_1$ be the order in which the numbers appear in V. We construct a partial 1-sketch as follows:

Place $\alpha^{(0)}_{a_1}, \alpha^{(0)}_{a_2}, \ldots, \alpha^{(0)}_{a_n}$ in order. Then insert some β-letters as follows: For each $i \in [n]$,

1. If there does not exist some $j > i$ such that $a_j < a_i$, we do not place $\alpha^{(1)}_{a_i}$.
2. If there exists some $j > i$ such that $a_j < a_i$, we place $\alpha^{(1)}_{a_i}$ in the space between $\alpha^{(0)}_{a_l}$ and the α-letter following it where l is the largest $l > i$ such an arc drawn between a_l and a_i does not contain an arc of V (no requirement of $a_l < a_i$) and if no such l exists, we place $\alpha^{(1)}_{a_i}$ between $\alpha^{(0)}_{a_i}$ and the α-letter following it.

Remark 3.2. Since the order of the α and β letters is the same, if there is more that one β-letter between two α-letters, there is a unique way to order them.

Example 3.8. Consider the valid arc diagram in Figure 3.9.

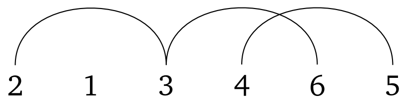


Figure 3.9: Example of a valid arc diagram.

Placing the $\alpha^{(1)}_{a_i}$ for which there exists $j > i$ such that $a_j < a_i$:

$$\boldsymbol{\alpha^{(0)}_5} \quad \boldsymbol{\alpha^{(0)}_6}\ \alpha^{(1)}_5 \quad \boldsymbol{\alpha^{(0)}_4}\ \alpha^{(1)}_6 \quad \boldsymbol{\alpha^{(0)}_3} \quad \boldsymbol{\alpha^{(0)}_1}\ \alpha^{(1)}_4\ \alpha^{(1)}_3\ \boldsymbol{\alpha^{(0)}_2}$$

Completing the placement (this is not the only way):

$$\alpha^{(0)}_5\ \alpha^{(0)}_6\ \alpha^{(1)}_5\ \alpha^{(0)}_4\ \alpha^{(1)}_6\ \alpha^{(0)}_3\ \alpha^{(0)}_1\ \alpha^{(1)}_4\ \alpha^{(1)}_3\ \alpha^{(1)}_1\ \alpha^{(0)}_2\ \alpha^{(1)}_2$$

Hence the Catalan region corresponding to this 1-sketch has points that satisfy all the inequalities specified by the arc diagram.

Given a valid arc diagram of length n, we associate a parking function of length n to it as follows:

From the valid arc diagram, we obtain a partition of $[n]$ whose blocks are the numbers that are joined by arc chains (see example below). For each $i \in [n]$ define $f(i)$ as the position of leftmost element in block containing i. Since the first i numbers in the diagram will have f value less than or equal to i, f is a parking function (we think of f as $(f(1), \ldots, f(n))$).

Example 3.9. The partition for the valid arc diagram of Figure 3.10 is $\{\{2,4,6\},\{1,5\},\{3\}\}$. The parking function is (3,1,6,1,3,1).

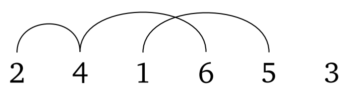


Figure 3.10: Example of a valid arc diagram.

To show that this association is a bijection, we will prove that for any parking function of length n there is exactly one valid arc diagram of length n for which it is the associated parking function. We do this by constructing all possible valid arc diagrams that can have it as the associated parking function and show that there is exactly one way to do this.

Let f be a parking function of length n. Any valid arc diagram associated to it (if such exists) must have the elements of $f^{-1}(i)$ forming a maximal arc chain for each $i \in [n]$ such that $f^{-1}(i) \neq \phi$. Since valid arc diagrams can only have arcs from smaller to larger numbers, the elements of $f^{-1}(i)$ must appear in ascending order in such a valid arc diagram. Also, the maximal chain corresponding to $f^{-1}(i)$ must have its first element in the i^{th} position.

So, from the previous paragraph, we just have to show that there is exactly one way to place these maximal chains (respecting positions of the first element of each chain) to get a valid arc diagram. We do so by placing the maximal chains in increasing order of f-value. Before going ahead with the proof, let us look at an example.

Example 3.10. Consider the parking function (4,2,1,1,1,4,2).

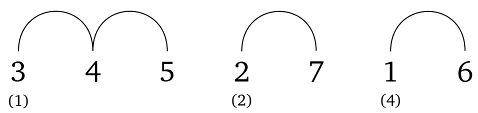


Figure 3.11: Maximal chains corresponding to $(4, 2, 1, 1, 1, 4, 2)$.

Figure 3.11 shows the maximal chains corresponding to the given parking function. The numbers in brackets represent the position of the first number in each maximal chain, that is, the f-value of the maximal chain. So placing these maximal chains in increasing order of f-value, we can see that there is exactly one way to do it to avoid chain containment.

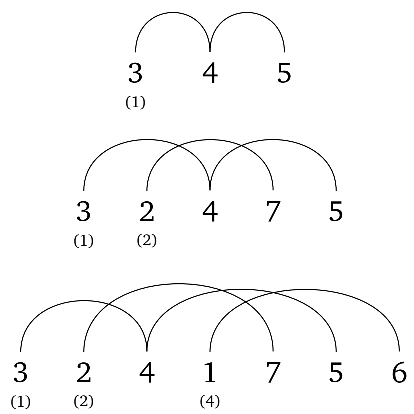


Figure 3.12: Placing maximal chains in increasing order of f-value.

We will call the number immediately before i in the maximal chain containing i the predecessor of i. In the above example, the predecessor of 5 is 4. So the predecessor of a number is defined if it is not the first element of the maximal chain.

We will now proceed with the proof, which can be found in [12]. Start with the empty diagram. Since f is a parking function, $f^{-1}(1) \neq \phi$. Place the maximal chain with f-value 1 in the empty diagram (there is only one way to do this). We will proceed by induction. Suppose all maximal chains with f-value $< j$ have been placed so that the diagram has no arc containing another and the position of the first element of any maximal chain is its corresponding f-value.

If $f^{-1}(j) = \phi$, there is nothing to prove. Suppose $f^{-1}(j) \neq \phi$. Let $f^{-1}(j) = \{i_1, \ldots, i_k\}$ where $i_1 < \cdots < i_k$. Since f is a parking function, there are at least $j-1$ elements with f-value less than or equal to $j-1$. Hence, there are at least $j-1$ numbers in the diagram (since all maximal chains with f-value less than or equal to $j-1$ have been placed). So there is a unique place where we should keep i_1 in the diagram so that it is in the j^{th} position. This logic is valid because we are placing chains in increasing order of f-value, so no element will be placed before i_1 and hence it will remain in the j^{th} position till the end.

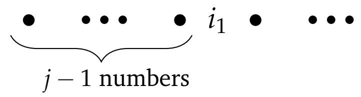


Figure 3.13: At least $j-1$ numbers before i_1.

We will use induction again. Suppose $\{i_1, \ldots, i_m\}$ have been placed uniquely. We will show that there is a unique way to place i_{m+1}. Once we do this, our proof will be complete. Remember that i_{m+1} should be placed somewhere after i_m, so if there are no numbers after i_m, we are done. Suppose there are numbers after i_m.

The numbers are after i_m are from maximal chains having f-value less than or equal to $j-1$. Hence each such number will have an arc chain joining it to some number before i_m (this arc chain may have more than one arc). So the predecessor of any such number is well-defined.

If all numbers to the right of i_m have predecessors to the left of i_m, i_{m+1} should be placed at the end. Or else we would have some arc containing another.

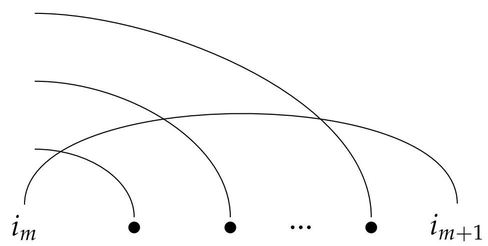


Figure 3.14: All numbers to the right of i_m having predecessors to the left of i_m.

Next, suppose there is some number to the right of i_m whose predecessor is also to the right of i_m. Let x be the leftmost such number, x' be the predecessor of x and y be the number before x. We will show that the only place that i_{m+1} can be placed such that no arc contains another is between y and x.

Suppose i_{m+1} is placed to the left of y. By choice of x, the predecessor of y is before i_m. Hence the arc from the predecessor of y to y will contain the arc from i_m to i_{m+1}.

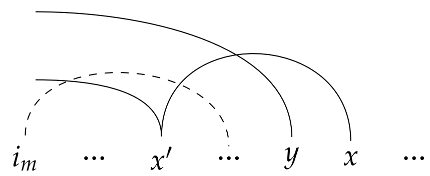


Figure 3.15: i_{m+1} being placed before y.

Suppose i_{m+1} is placed after x. In this case, the arc from x' to x will be contained in the arc from i_m to i_{m+1}.

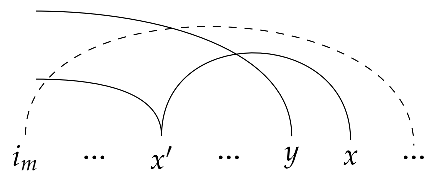


Figure 3.16: i_{m+1} being placed after x.

We must now show that no arc is contained in another if we place i_{m+1} between y and x. Suppose this is false. Then, by the induction hypothesis, either the arc from i_m to i_{m+1} contains an arc, which would mean there is an element before x but after i_m that has a predecessor to the right of i_m (contradiction to choice of x). Or the arc from i_m to i_{m+1} is contained in some arc, which would mean that this arc also contains the arc from x' to x (contradiction to induction hypothesis). In either case, we get a contradiction. Hence the only way we can place i_{m+1} is between y and x.

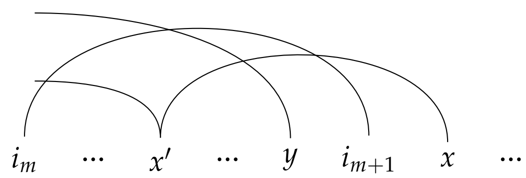


Figure 3.17: i_{m+1} being placed between y and x.

Figure 3.18 shows the labelling of the regions of $\mathcal{S}_3$ using parking functions. This is usually called the Athanasiadis-Linusson labelling.

The methods described here can be extended to obtain a bijection between regions of the m-Shi regions in $\mathbb{R}^n$ and m-parking functions of length n. We define m-parking functions and refer the reader to [4] for the bijection. Notice that 1-parking functions are the usual parking functions.

Definition 3.4 (m-parking functions of length n)**.** A sequence $(a_1, \ldots, a_n)$ of positive integers such that if $(b_1, \ldots, b_n)$ is the sequence in ascending order, we get $b_i \leq 1 + m(i-1)$ for all $i \in [n]$.

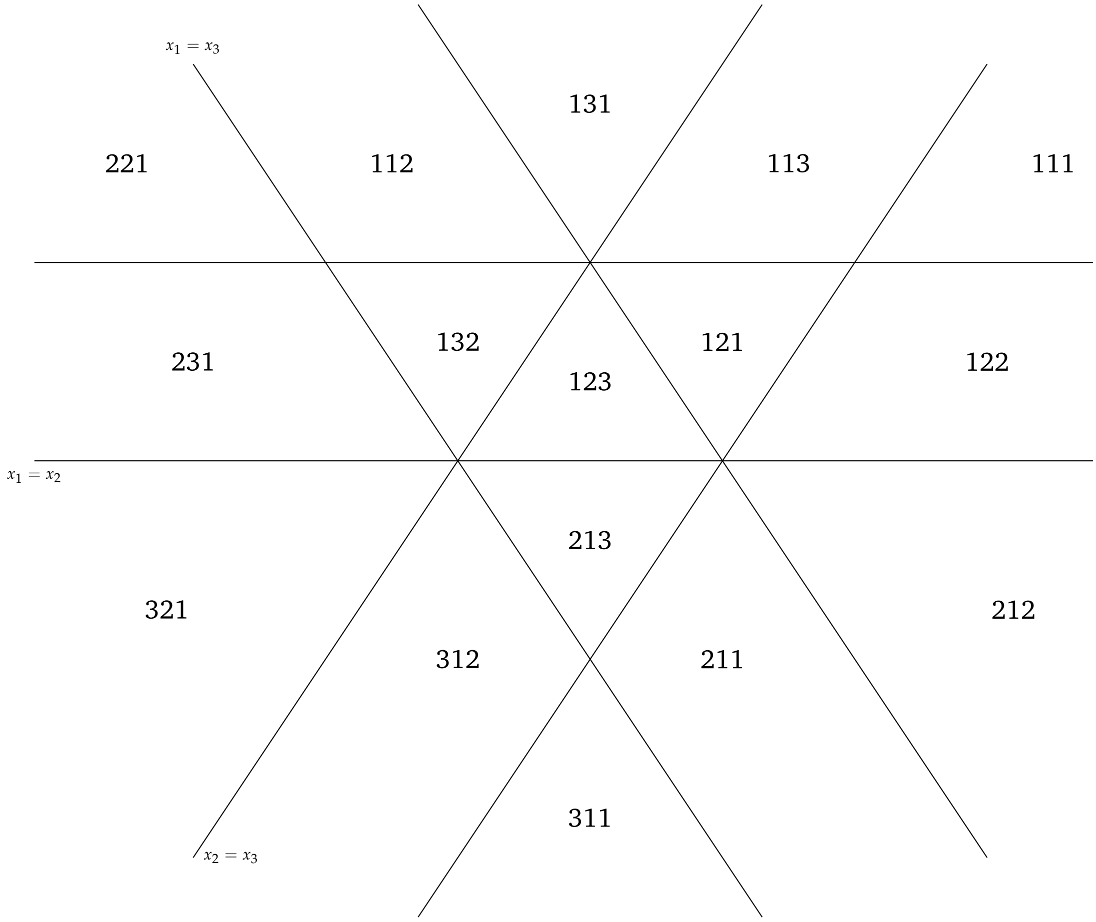


Figure 3.18: Athanasiadis-Linusson labelling of regions of $\mathcal{S}_3$.

Chapter 4

Sketches and trees

In his paper [5], Bernardi describes a method to count the regions of any deformation of the braid arrangement using certain objects called *boxed trees*. For certain deformations, which he calls *transitive* deformations, he obtains an explicit bijection between the regions and a certain set of trees. In this chapter, we will describe this bijection for the special case of Catalan, Shi and Linial arrangements, which are all transitive deformations.

4.1 Trees

We will first state some definitions and results from the paper [5].

Definition 4.1 (Rooted tree)**.** A tree is a graph with no cycles. A rooted tree is a tree with a distinguished vertex called the root.

We will draw rooted trees with their root at the bottom. Children of a vertex v in a rooted tree are those vertices w that are adjacent to v and such that the unique path from the root to w passes through v. Similarly we can define the parent of a vertex v to be the vertex w for which it v is the child of w. Any non-root vertex has a unique parent. All the vertices that have at least one child are called nodes and those that do not are called leaves. The children of any node will be ordered from left to right. Hence the left siblings of a vertex v are the vertices that are also children of the parent of v but are to the left of v. We denote the number of left siblings of v as $\text{lsib}(v)$. Example 4.1 might help clear up any doubts about the definitions.

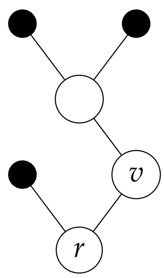


Figure 4.1: Example of a rooted tree.

Example 4.1. The vertex r of Figure 4.1 is the root. It has a leaf and the vertex v as its children and this leaf is the only left sibling of v. So $\text{lsib}(v) = 1$. The nodes are white while the leaves are black.

Definition 4.2 ((m+1)-ary tree)**.** A rooted tree where each node has exactly $m + 1$ children.

Definition 4.3. We will denote by $\mathcal{T}^{(m)}(n)$ the set of all $(m + 1)$-ary trees with n nodes labeled with distinct elements from $[n]$.

Example 4.2. Figure 4.2 shows an element of $\mathcal{T}^{(1)}(3)$.

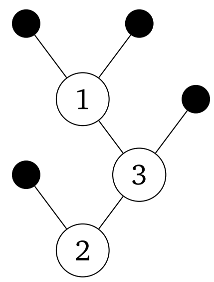


Figure 4.2: An element of $\mathcal{T}^{(1)}(3)$

For trees in $\mathcal{T}^{(m)}(n)$, we will denote the node having label $i \in [n]$ by just i.

Definition 4.4. If a node i in a tree $T \in \mathcal{T}^{(m)}(n)$ has at least one child that is a node, we denote by $\text{cadet}(i)$ the rightmost such child.

Definition 4.5. For any finite set of integers S, define $\mathcal{T}_S(n)$ to be the set of trees in $\mathcal{T}^{(m)}(n)$, where $m = \max \{|s| \mid s \in S\}$, such that if $\text{cadet}(i) = j$:

- $\text{lsib}(j) \notin S \cup \{0\} \Rightarrow i < j$.
- $-\text{lsib}(j) \notin S \Rightarrow i > j$.

Recall that for any finite set of integers S, we defined the arrangement $\mathcal{A}_S(n)$ as the deformation of the braid arrangement in $\mathbb{R}^n$ with hyperplanes

$$\{x_i - x_j = k \mid k \in S, \quad 1 \le i < j \le n\}.$$

Though Bernardi derived results for more general deformations, we will only be focused on these.

Definition 4.6. A finite set of integers S is said to be transitive if for any $s, t \notin S$,

- $st > 0 \Rightarrow s + t \notin S$.
- $s > 0$ and $t \le 0 \Rightarrow s - t \notin S$ and $t - s \notin S$.

We can now state the result for arrangements $\mathcal{A}_S(n)$ where S is transitive.

Theorem 4.1. *For any transitive set of integers S, the regions of the arrangement $\mathcal{A}_S(n)$ are in bijection with the trees in $\mathcal{T}_S(n)$.*

Recall that the Catalan, Shi and Linial arrangements in $\mathbb{R}^n$ are $\mathcal{A}_{\{-1,0,1\}}(n)$, $\mathcal{A}_{\{0,1\}}(n)$ and $\mathcal{A}_{\{1\}}(n)$ respectively. It can be checked that $\{-1,0,1\}$, $\{0,1\}$ and $\{1\}$ are transitive sets. Also, for all these arrangements, the corresponding sets $\mathcal{T}_S(n)$ is a subset of $\mathcal{T}^{(1)}(n)$ and for the Catalan arrangement, that is $S = \{-1,0,1\}$, $\mathcal{T}_{\{-1,0,1\}}(n) = \mathcal{T}^{(1)}(n)$.

Similarly, it can be shown that $\mathcal{T}_{\{0,1\}}(n)$ is the set of trees in $\mathcal{T}^{(1)}(n)$ such that if the right child of node i is the node j then $i > j$.

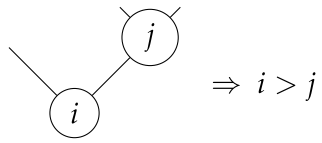


Figure 4.3: Condition for an element of $\mathcal{T}^{(1)}(n)$ to belong to $\mathcal{T}_{\{0,1\}}(n)$.

Also, $\mathcal{T}_{\{1\}}(n)$ is the set of trees in $\mathcal{T}^{(1)}(n)$ such that:

1. If the right child of node i is the node j then $i > j$.
2. If the left child of node i is the node j and its right child is a leaf then $i > j$.

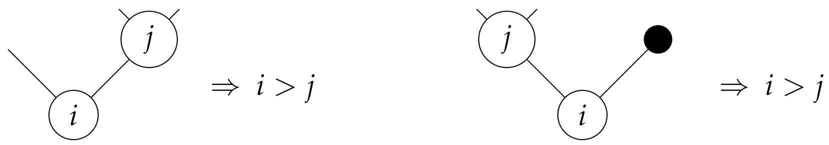


Figure 4.4: Conditions for an element of $\mathcal{T}^{(1)}(n)$ to belong to $\mathcal{T}_{\{1\}}(n)$.

Hence the idea for the bijections is as follows: We will first obtain the bijection of the regions of the Catalan arrangement $\mathcal{A}_{\{-1,0,1\}}(n)$ with $\mathcal{T}^{(1)}(n)$. Since each Catalan region is in some Shi (respectively Linial) region and each Shi (respectively Linial) region contains some Catalan region, we will choose a canonical Catalan region in each Shi (respectively Linial) region. These choices will induce the desired bijections for the Shi and Linial arrangements.

The constructions and bijections will be done for the usual Catalan, Shi and Linial arrangements. However, these can also be extended to the m-Catalan, m-Shi and m-Linial arrangements.

4.2 The Catalan arrangement

We will first recall the bijection for Catalan regions from Section 3.2.

Any region of $\mathscr{C}_n$ is given by a total order on the symbols $x_1, \ldots, x_n, x_1+1, \ldots, x_n+1$ and this is represented by a word in $A^{(1)}(n) = \{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\}$ such that:

1. Each letter in $A^{(1)}(n)$ appears exactly once.

2. For any $i, j \in [n]$, $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)} \Rightarrow \alpha_i^{(1)}$ appears before $\alpha_j^{(1)}$.

3. For any $i \in [n]$, $\alpha_i^{(0)}$ appears before $\alpha_i^{(1)}$.

Such words are called 1-sketches. Here, $\alpha_i^{(s)}$ represents $x_i + s$ and the order of the letters represents the order on the symbols $x_1, \ldots, x_n, x_1 + 1, \ldots, x_n + 1$.

We will denote the set of 1-sketches by $\mathcal{D}^{(1)}(n)$. Hence our objective is to obtain a bijection from $\mathcal{D}^{(1)}(n)$ to $\mathcal{T}^{(1)}(n)$. We will obtain inverse maps $\phi_1 : \mathcal{D}^{(1)}(n) \to \mathcal{T}^{(1)}(n)$ and $\psi_1 : \mathcal{T}^{(1)}(n) \to \mathcal{D}^{(1)}(n)$. The subscripts 1 are to indicate that there are similar bijections between $\mathcal{T}^{(m)}(n)$ and $\mathcal{D}^{(m)}(n)$, the set of m-sketches defined in Section 3.2.

Before going ahead with the description of these maps, we need to introduce some terminology associated to trees.

For any tree rooted tree T, the drift of a vertex v in T is defined as:

$$\text{drift}(v) = \text{lsib}(u_0) + \text{lsib}(u_1) + \cdots + \text{lsib}(u_k)$$

where u_0 is the root of T and $u_0, u_1, \ldots, u_k = v$ is the path from the root to v. Note that $\text{lsib}(u_0) = 0$.

For any rooted tree T, we define the *postfix* order on the vertices, which we will denote by $<_{pf}$, as the order in which the vertices of T occur when traversing the tree in the counter-clockwise direction, starting from the root. A more formal definition can be found in [22].

Example 4.3. For the rooted tree given in Figure 4.5,

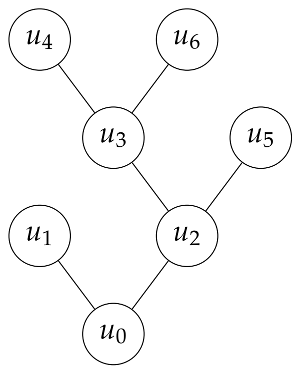


Figure 4.5: Example of a rooted tree.

- $\text{drift}(v) = 0$ for $v = u_0, u_1$.
- $\text{drift}(v) = 1$ for $v = u_2, u_3, u_4$.
- $\text{drift}(v) = 2$ for $v = u_5, u_6$.
- $u_0 <_{pf} u_2 <_{pf} u_5 <_{pf} u_3 <_{pf} u_6 <_{pf} u_4 <_{pf} u_1$.

For any rooted tree T, we define a total order $<_T$ on the vertices as follows: $u <_T v$ if $\text{drift}(u) < \text{drift}(v)$ or $\text{drift}(u) = \text{drift}(v)$ and $u <_{pf} v$. That is, we first order them by drift and then order those with equal drift in $<_{pf}$ order. So in the example above $u_0 <_T u_1 <_T u_2 <_T u_3 <_T u_4 <_T u_5 <_T u_6$.

We now describe the map $\phi_1 : \mathcal{D}^{(1)}(n) \to \mathcal{T}^{(1)}(n)$. For each word in $w \in \mathcal{D}^{(1)}(n)$, we will construct the tree $\phi_1(w)$ via intermediate trees called budding trees. These are trees with at most n nodes labeled distinctly with elements from $[n]$ and a special set of leaves called buds. For any budding tree T, we will call the first bud of T the least bud in the $<_T$ order.

Let $w \in \mathcal{D}^{(1)}(n)$ and let $w = w_1 \dots w_{2n}$. Define $T_0(w)$ to be the budding tree with just one bud and no other vertex. For $p \in [2n]$, $T_p(w)$ is the tree obtained from $T_{p-1}(w)$ by replacing its first bud by

- a leaf if w_p is a β-letter (that is, $\alpha_i^{(1)}$ for some i).
- a node labeled i with 2 bud children if $w_p = \alpha_i^{(0)}$.

Finally, we get $\phi_1(w)$ by replacing the bud of $T_{2n}(w)$ (it has exactly one bud) by a leaf. It can be checked that $T_p(w)$ has $1 + n_\alpha - n_\beta$ buds where n_α is the number of α-letters in $w_1 \dots w_p$ and n_β is the number of β-letters in $w_1 \dots w_p$. Since w is a 1-sketch, this will mean that $T_p(w)$ always has at least one bud (property 2 of Proposition 3.2). It can also be checked that $\phi_1(w) \in \mathcal{T}^{(1)}(n)$ and hence ϕ_1 is well-defined.

Example 4.4. Consider the word $w = \alpha_2^{(0)}\alpha_2^{(1)}\alpha_3^{(0)}\alpha_1^{(0)}\alpha_3^{(1)}\alpha_1^{(1)}$ in $\mathcal{D}^{(1)}(3)$. The construction of $\phi_1(w)$ is given in Figure 4.6 (square vertices represent buds).

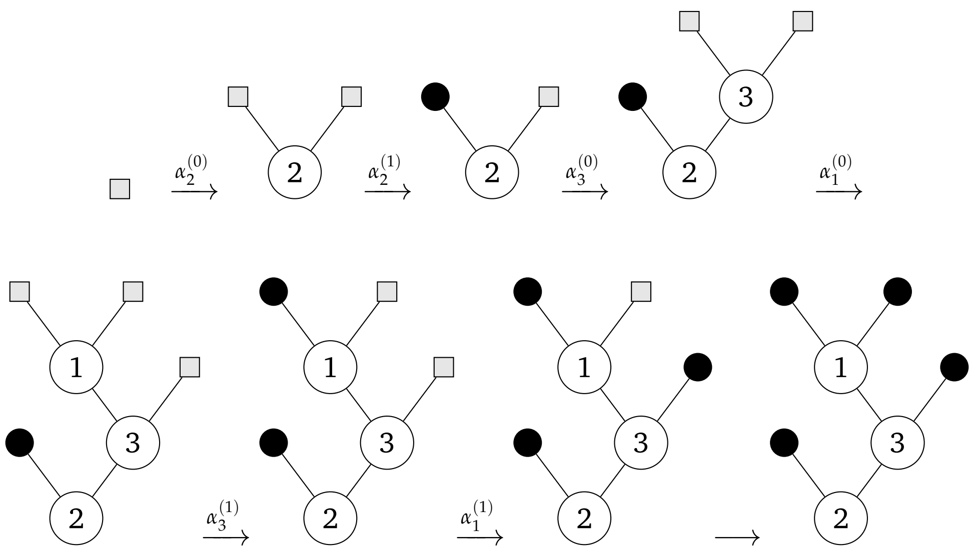


Figure 4.6: Construction of $\phi_1(\alpha_2^{(0)}\alpha_2^{(1)}\alpha_3^{(0)}\alpha_1^{(0)}\alpha_3^{(1)}\alpha_1^{(1)})$.

Before we proceed with describing the map ψ_1, we note the following lemma.

Lemma 4.1. *For a rooted tree T, the successor of a node in the $<_T$ order is its first child.*

Proof. Let u be a node of T. First we note that the first child of u, which we call v, has the same drift as u and $u <_{pf} v$. Hence $u <_T v$. Since v has the same drift as u, we just have to show that any vertex w of T such that $u <_{pf} w <_{pf} v$ has drift greater than that of u. But this is true because such w are from the subtrees $T_1, \ldots, T_k$ which have roots as the right siblings of v (see Figure 4.7).

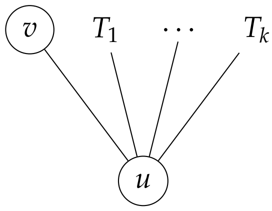


Figure 4.7: Vertices between a node and its first child in the postfix order.

□

Let $w = w_1 \ldots w_{2n}$ be a word in $\mathcal{D}^{(1)}(n)$. Let $T = \phi_1(w)$ and $v(w_p)$ be the vertex created corresponding to w_p, that is the non-bud created in $T_p(w)$. From the construction of T and the above lemma, it can be seen that

$$v(w_1) <_T \cdots <_T v(w_{2n}) <_T l \tag{4.1}$$

where l is the leaf substituted for the bud in $T_{2n}(w)$. We use this observation to construct the inverse.

Before doing so we recall that a word in $\mathcal{D}^{(1)}(n)$ can be represented as a word with letters $\alpha_1, \ldots, \alpha_n$ and n β's (Proposition 3.2). We will use this notation to describe ψ_1. To avoid confusion, we will denote this representation of $\psi_1(T)$ for any $T \in \mathcal{T}^{(1)}(n)$ as $\tilde{\psi_1}(T)$.

Let $T \in \mathcal{T}^{(1)}(n)$. Also, let

$$u_0 <_T \cdots <_T u_{2n}$$

be the $<_T$ order on the vertices of T. We define $\tilde{\psi_1}(T)$ to be $\tilde{w_1} \ldots \tilde{w_{2n}}$ where

$$\tilde{w_p} = \begin{cases} \alpha_i, & \text{if } u_{p-1} \text{ is the node } i \\ \beta, & \text{if } u_{p-1} \text{ is a leaf} \end{cases}$$

for any $p \in [2n]$. Let $\psi_1(T)$ be the corresponding word in $\mathcal{D}^{(1)}(n)$. Using the fact that any node in a tree T has 2 children, and parent $<_T$ child, it can be shown that $\psi_1(T)$ is indeed a 1-sketch.

By (4.1), we can see that $\psi_1 \circ \phi_1$ is the identity map on $\mathcal{D}^{(1)}(n)$. It is also known that $\mathcal{T}^{(1)}(n)$ has $n! \times \frac{1}{n+1}\binom{2n}{n}$ elements (refer [22]), which is the same as $\mathcal{D}^{(1)}(n)$. Hence ϕ_1 and ψ_1 are bijections.

We will now mention another way to describe ψ_1 which will help in the bijections for the Shi and Linial arrangements.

Let $T \in \mathcal{T}^{(1)}(n)$ and $u_0 <_T \cdots <_T u_{2n}$ be the vertices of T. Let $w = w_1 \ldots w_{2n}$ where $w_p = \alpha_i^{(s)}$ if u_p is the $(s+1)^{th}$ child of the node i. We claim that $\psi_1(T) = w$. Since any non-root vertex is the child of some node, the definition of w makes sense. It can be checked that w is a 1-sketch. Also by the lemma above, we know that u_p is the first child of the node i if and only if

u_{p-1} is the node i. So $w_p = \alpha_i^{(0)}$ if and only if u_{p-1} is the node i. So w corresponds to the word $\tilde{\psi_1}(T)$ and hence $\psi_1(T) = w$.

Using both descriptions of ψ_1, we obtain the following lemma.

Lemma 4.2. *Let w be a 1-sketch. The $(s+1)^{th}$ child of node i in $\phi_1(w)$ is*

- *the node j if and only if the letter immediately following $\alpha_i^{(s)}$ is $\alpha_j^{(0)}$.*
- *a leaf if and only if the letter immediately following $\alpha_i^{(s)}$ is a β-letter.*

Combining the bijection of $\mathcal{T}^{(1)}(n)$ with $\mathcal{D}^{(1)}(n)$ and that of $\mathcal{D}^{(1)}(n)$ with the regions of $\mathscr{C}_n$, we get the bijections Φ_1 from the regions of $\mathscr{C}_n$ to $\mathcal{T}^{(1)}(n)$ and its inverse Ψ_1. We will now describe Ψ_1.

For any $T \in \mathcal{T}^{(1)}(n)$, the region $\Psi_1(T)$ of $\mathscr{C}_n$ is the region defined by

$$x_i - x_j < s, \text{ if } i <_T v \text{ where } v \text{ is the } (s+1)^{th} \text{ child of node } j$$
$$x_i - x_j > s, \text{ otherwise}$$

for $s \in \{0, 1\}$ and distinct $i, j \in [n]$.

Note that since $i, j \in [n]$ are distinct, the first child of i being less than the $(s+1)^{th}$ child of j in the $<_T$ order is the same as the node i being less that the $(s+1)^{th}$ child of j. This is because the node i is the predecessor of its first child in the $<_T$ order and also because a child of i cannot also be a child of j.

4.3 The Shi arrangement

Each Shi region contains some Catalan regions. Our goal is to choose a canonical Catalan region for each Shi region which will then induce the desired bijection between regions of $\mathcal{S}_n$ and $\mathcal{T}_{\{0,1\}}(n)$. It will be convenient to think of 1-sketches, trees in $\mathcal{T}^{(1)}(n)$ and regions of $\mathscr{C}_n$ interchangeably. The method used will be to order the 1-sketches and choose the maximum 1-sketch in this order from each Shi region to be its canonical representative.

First, we need to partition the 1-sketches based on the Shi region they lie in. To do so, we make the following definition.

Definition 4.7 (Shi move)**.** Swapping two consecutive letters of the form $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$ where $i < j$ in a 1-sketch is called a Shi move.

Example 4.5. Applying a Shi move to the 1-sketch $\alpha_1^{(0)}\alpha_1^{(1)}\alpha_2^{(0)}\alpha_3^{(0)}\alpha_2^{(1)}\alpha_3^{(1)}$ we can obtain the following:

- $\alpha_1^{(0)}\alpha_2^{(0)}\alpha_1^{(1)}\alpha_3^{(0)}\alpha_2^{(1)}\alpha_3^{(1)}$ (swapping $\alpha_1^{(1)}$ and $\alpha_2^{(0)}$).
- $\alpha_1^{(0)}\alpha_1^{(1)}\alpha_2^{(0)}\alpha_2^{(1)}\alpha_3^{(0)}\alpha_3^{(1)}$ (swapping $\alpha_2^{(1)}$ and $\alpha_3^{(0)}$).

Remark 4.1. It can be checked that performing a Shi move on a 1-sketch results in a 1-sketch. Note that swapping consecutive $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$ where $i < j$ in a 1-sketch has the effect, in the corresponding Catalan regions, of changing the inequality between $x_i - x_j$ and -1 while all other inequalities remain the same.

Definition 4.8 (Shi equivalent)**.** Two 1-sketches are said to be Shi equivalent if one can be obtained from the other by performing a series of Shi moves.

Lemma 4.3. *Two 1-sketches are Shi equivalent if and only if their corresponding Catalan regions lie in the same Shi region.*

Proof. By the above remark, two Shi equivalent regions lie on the same side of $x_i - x_j = s$ hyperplanes for $i < j$ and $s = 0, 1$. Hence they lie in the same Shi region. Conversely, let two Catalan regions R_1, R_2 lie in the same Shi region. If $R_1 = R_2$, we are done. If not, then R_2 lies on the opposite side as R_1 of at least one of the bounding hyperplanes of R_1 (bounding hyperplane is one such that just changing its inequality and keeping the others the same results in a nonempty region). Using this and induction, we can get that R_2 can be obtained from R_1 by changing exactly one inequality of the type $x_i - x_j > -1$ or $x_i - x_j < -1$ at a time (where $i < j$). This corresponds to a series of Shi moves. Hence R_1 and R_2 are Shi equivalent. □

Next, we need to order the 1-sketches. We will use the lexicographic order induced by the following order on the letters:

$$\alpha_1^{(1)} \prec \alpha_2^{(1)} \prec \cdots \prec \alpha_n^{(1)} \prec \alpha_1^{(0)} \prec \alpha_2^{(0)} \prec \cdots \prec \alpha_n^{(0)}.$$

This means that for two distinct 1-sketches w, w', $w \prec w'$ if and only if the first different letter is lesser in w (that is, if $w = w_1 \dots w_{2n}$ and $w' = w'_1 \dots w'_{2n}$ and $w_k = w'_k$ for all $k \in [p-1]$ and $w_p \neq w'_p$ and $w_p \prec w'_p$).

Definition 4.9 (Shi maximal)**.** A 1-sketch is said to be Shi maximal if it is greater than all 1-sketches that are Shi equivalent to it.

Hence, using the lemma above, Shi regions are in bijection with Shi maximal 1-sketches. However, it is not easy to describe these Shi maximal 1-sketches directly. It turns out that to check whether a 1-sketch is Shi maximal, we only need to compare it with the 1-sketches that are obtained by applying a single Shi move to it.

Definition 4.10 (Shi locally maximal)**.** A 1-sketch is said to be Shi locally maximal if it is greater than all 1-sketches that can be obtained by applying a single Shi move to it.

So we have to show that a 1-sketch is Shi maximal if and only if it is Shi locally maximal. Note that Shi locally maximal 1-sketches are easier to describe. A 1-sketch is Shi locally maximal if and only if there is no $i < j$ such that $\alpha_i^{(1)}$ that is immediately followed by $\alpha_j^{(0)}$. In fact, such 1-sketches are precisely the ones that correspond to trees in $\mathcal{T}_{\{0,1\}}(n)$. Once we prove these two facts, we will obtain the required bijection between regions of $\mathcal{S}_n$ and $\mathcal{T}_{\{0,1\}}(n)$ induced by Ψ_1 which we call $\Psi_{\{0,1\}}$:
For any $T \in \mathcal{T}_{\{0,1\}}(n)$, the region $\Psi_{\{0,1\}}(T)$ of $\mathcal{S}_n$ is the region defined by

$$x_i - x_j < 0, \text{ if and only if } i <_T j$$
$$x_i - x_j < 1, \text{ if and only if } i <_T v \text{ where } v \text{ is the right child of } j$$

for all $1 \leq i < j \leq n$.

So it remains to prove the following lemma.

Lemma 4.4. *Let w be a 1-sketch. The following are equivalent:*

1. *w is Shi maximal.*
2. *w is Shi locally maximal.*
3. *The tree corresponding to w, that is, $\phi_1(w)$ is in $\mathcal{T}_{\{0,1\}}(n)$.*

Proof. We will first show $1 \Leftrightarrow 2$.

It is clear that $1 \Rightarrow 2$. For the other implication we proceed by contradiction. Suppose w is Shi locally maximal but not maximal. That is, there is some w' that is Shi equivalent to w such that $w \prec w'$. Let $w = w_1 \dots w_{2n}$ and $w' = w'_1 \dots w'_{2n}$. Since $w \prec w'$, we have some $p \in [2n]$ such that $w_k = w'_k$ for all $k \in [p-1]$ and $w_p \prec w'_p$.

Since w and w' are Shi equivalent, the α-letters (the (0)-type letters) and hence the β-letters ((1)-type) appear in the same order in both (this is because their corresponding regions both lie on the same side of $x_i - x_j = 0$ for all distinct $i, j \in [n]$). Hence both w_p and w'_p cannot be of the same type, that is, one should be an α-letter and the other should be a β-letter. Since $w_p \prec w'_p$, we must have $w_p = \alpha_i^{(1)}$ and $w'_p = \alpha_j^{(0)}$ for some $i, j \in [n]$.

Again since the order of the α-letters is the same in both w and w', the first α-letter after w_p in w has to be $\alpha_j^{(0)}$. Let $w_q = \alpha_j^{(0)}$ where $q > p$. Now $w_{q-1} = \alpha_k^{(1)}$ for some $k \in [n]$. Since $w_q = \alpha_j^{(0)}$, we cannot have $k = j$.

If $k > j$, then $\alpha_k^{(1)}$ must be before $\alpha_j^{(0)}$ in w' as well (since both regions corresponding to w and w' lie on the same side of $x_j - x_k = 1$). But this would mean $\alpha_k^{(1)}$ is in $w'_1 \dots w'_{p-1} = w_1 \dots w_{p-1}$, which is a contradiction since $w_{q-1} = \alpha_k^{(1)}$ and $q - 1 \geq p$.

If $k < j$, then w is not Shi locally maximal since we could apply the Shi move of swapping $\alpha_k^{(1)}$ and $\alpha_j^{(0)}$ to obtain a greater 1-sketch. Hence w must be Shi maximal.

We will now show $2 \Leftrightarrow 3$.

w is Shi locally maximal if and only if there is no $i < j$ such that $\alpha_i^{(1)}$ that is immediately followed by $\alpha_j^{(0)}$. Translating this in terms of the tree $\phi_1(w)$, we get that $\phi_1(w)$ should not have $i < j$ with the right child of a node i being the node j. But these are precisely the trees in $\mathcal{T}_{\{0,1\}}(n)$. Hence we get that w is Shi locally maximal if and only if $\phi_1(w)$ is in $\mathcal{T}_{\{0,1\}}(n)$. □

4.4 The Linial arrangement

The bijection between regions of $\mathscr{L}_n$ and the $\mathcal{T}_{\{1\}}(n)$ is obtained using the same methods as for the Shi arrangement. The main difference is in the proof that Linial locally maximal sketches are Linial maximal (definitions given below).

We will now state the corresponding definitions and lemmas for the Linial arrangement. The ordering of the 1-sketches is the same as before, that is, the lexicographic order induced by:

$$\alpha_1^{(1)} \prec \alpha_2^{(1)} \prec \cdots \prec \alpha_n^{(1)} \prec \alpha_1^{(0)} \prec \alpha_2^{(0)} \prec \cdots \prec \alpha_n^{(0)}.$$

Definition 4.11 (Linial move)**.** A Linial move on a 1-sketch is performing either one of the following:

- Swapping two consecutive letters of the form $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$ where $i < j$ (Shi move).
- Swapping the letters $\alpha_i^{(0)}$ and $\alpha_j^{(0)}$ and also swapping $\alpha_i^{(1)}$ and $\alpha_j^{(1)}$ where both pairs are consecutive in the 1-sketch.

Remark 4.2. It can be checked that performing a Linial move on a 1-sketch results in a 1-sketch. The first type of Linial move correspond to changing the inequality between $x_i - x_j$ and -1 while all other inequalities remain the same. While the second type corresponds to changing the inequality between $x_i - x_j$ and 0 while all other inequalities remain the same.

Definition 4.12 (Linial equivalent)**.** Two 1-sketches are said to be Linial equivalent if one can be obtained from the other by performing a series of Linial moves.

Lemma 4.5. *Two 1-sketches are Linial equivalent if and only if their corresponding Catalan regions lie in the same Linial region.*

Definition 4.13 (Linial maximal)**.** A 1-sketch is said to be Linial maximal if it is greater than all 1-sketches that are Linial equivalent to it.

Definition 4.14 (Linial locally maximal)**.** A 1-sketch is said to be Linial locally maximal if it is greater than all 1-sketches that can be obtained by applying a single Linial move to it.

The main lemma we need to prove to obtain the bijection is the following:

Lemma 4.6. *Let w be a 1-sketch. The following are equivalent:*

1. *w is Linial maximal.*
2. *w is Linial locally maximal.*
3. *The tree corresponding to w, that is, $\phi_1(w)$ is in $\mathcal{T}_{\{1\}}(n)$.*

Once this lemma is proved, we will obtain the required bijection between regions of $\mathscr{L}_n$ and $\mathcal{T}_{\{1\}}(n)$ induced by Ψ_1 which we call $\Psi_{\{1\}}$:
For any $T \in \mathcal{T}_{\{1\}}(n)$, the region $\Psi_{\{1\}}(T)$ of $\mathscr{L}_n$ is the region defined by

$$x_i - x_j < 1, \text{ if and only if } i <_T v \text{ where } v \text{ is the right child of } j$$

for all $1 \le i < j \le n$.

Proof of Lemma. We will first show $1 \Leftrightarrow 2$.

It is clear that $1 \Rightarrow 2$. For the other implication we proceed by contradiction. Suppose w is Linial locally maximal but not maximal. That is, there is some w' that is Linial equivalent to w such that $w \prec w'$. Let $w = w_1 \dots w_{2n}$ and $w' = w'_1 \dots w'_{2n}$. Since $w \prec w'$, we have some $p \in [2n]$ such that $w_k = w'_k$ for all $k \in [p-1]$ and $w_p \prec w'_p$.

If both w_p and w'_p are β-letters, they will be the same (the β corresponding to the $(k+1)^{th}$ α if there are k β-letters in $w_1 \dots w_{p-1}$). Hence either both are α-letters or one is an α and the other is a β. In either case, since $w_p \prec w'_p$, we must have that w'_p is an α-letter, say $\alpha_j^{(0)}$ for some $j \in [n]$. Suppose $w_{p+d} = \alpha_j^{(0)}$ where $d > 0$.

First we consider the case when all the letters $w_p, \dots, w_{p+d}$ are α-letters. Let $w_{p+k} = \alpha_{i_k}^{(0)}$. Since $i_0 < j = i_d$,taking i_c to be the minimum of all i_k we have $i_c < i_{c+1}$ and $i_c < j$. It can be shown, since w is Linial locally maximal, that the letter following $\alpha_{i_c}^{(1)}$ should be of the form $\alpha_k^{(0)}$ where $k < i_c$. Since w and w' are in the same Linial region and $\alpha_k^{(0)}$ is between $\alpha_{i_c}^{(1)}$ and $\alpha_j^{(1)}$ in w where $k < i_c < j$, the same is true in w'. But this gives that $\alpha_{i_c}^{(0)}$ is before $\alpha_j^{(0)}$ in w', which cannot happen since this would mean $\alpha_{i_c}^{(0)}$ appears in both $w_1 \dots w_{p-1}$ as well as $w_p \dots w_{p+d}$.

Next, suppose that one of the letters $w_p \dots w_{p+d}$ is a β-letter. Let $\alpha_i^{(1)}$ be the last β-letter before $w_{p+d} = \alpha_j^{(0)}$. Since w and w' lie in the same Linial region, $i < j$. Let the letter immediately after $\alpha_i^{(1)}$ be $\alpha_k^{(0)}$. Since w is Linial locally maximal, $k < i$. Hence we have that $k < j$ and all the letters between $\alpha_k^{(0)}$ and $\alpha_j^{(0)}$ are α-letters, and a similar argument as the previous paragraph leads to a contradiction. Hence w must be Linial maximal.

We will now show $2 \Leftrightarrow 3$.

w is Linial locally maximal if and only if

1. There is no $i < j$ such that $\alpha_i^{(1)}$ that is immediately followed by $\alpha_j^{(0)}$.

2. There is no $i < j$ such that $\alpha_i^{(0)}$ is followed by $\alpha_j^{(0)}$ and $\alpha_i^{(1)}$ is followed by $\alpha_j^{(1)}$.

Translating these conditions in terms of the tree $\phi_1(w)$, we get:

1. $\phi_1(w)$ should not have $i < j$ with the right child of a node i being the node j.

2. $\phi_1(w)$ should not have $i < j$ with the left child of a node i being the node j and the right child being a leaf.

But these are precisely the trees in $\mathcal{T}_{\{1\}}(n)$. Hence we get that w is Linial locally maximal if and only if $\phi_1(w)$ is in $\mathcal{T}_{\{1\}}(n)$. $\square$

Chapter 5

Deformations of type C

We have been looking at arrangements that have hyperplanes of the form $x_i - x_j = s$. Now we include hyperplanes of the form $x_i + x_j = s$ and $2x_i = s$ as well. Hyperplane arrangements of this form are called deformations of the type C arrangement. In this chapter we extend, with a modification inspired by [2], the results of [10] and count the number of regions of some deformations using the idea of 'moves' from Bernardi's paper [5].

5.1 Type C

The braid arrangement in $\mathbb{R}^n$ is the set of reflecting hyperplanes of the root system A_{n-1}. The type C arrangement in $\mathbb{R}^n$ is the set of reflecting hyperplanes of the root system C_n. Relevant definitions can be found in [8]. Its hyperplanes are:

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for $1 \leq i < j \leq n$. Before going forward with general deformations, we will first look at linear deformations. That is, sub-arrangements of the type C arrangement. Hence, in the spirit of Bernardi [5], we will define certain sketches corresponding to the region of the type C arrangement and for any deformation, we choose a canonical sketch from each region.

We can write the hyperplanes of the type C arrangement as follows:

$$\begin{aligned} x_i &= x_j, \quad && 1 \leq i < j \leq n \\ x_i &= -x_j, \quad && i, j \in [n]. \end{aligned}$$

Hence, any region of the arrangement is given by a valid total order on $x_1, \ldots, x_n, -x_1, \ldots, -x_n$.

We represent x_i by i and $-x_i$ by $-i$.

Example 5.1. The region $-x_2 < x_3 < x_1 < -x_1 < -x_3 < x_2$ is represented as $-2\ 3\ 1\ \ -1\ \ -3\ 2$.

It can be shown that words of the form:

$$i_1 \quad i_2 \quad \ldots \quad i_n \quad -i_n \quad \ldots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$ are the ones that correspond to regions. This is because negatives reverse order and also, choosing n distinct negative numbers, it is easy to construct a point satisfying the inequalities specified by such a word. Hence the number of regions of the type C arrangement is $2^n n!$. We will call such words 0-sketches. We will call the part of the sketch $i_1\ i_2\ \ldots\ i_n$ its first half and similarly the second half is the part $-i_n\ \ldots\ -i_2\ -i_1$. We will represent i by $\overset{+}{i}$ and $-i$ by $\overset{-}{i}$ for all $i \in [n]$ and draw a line between the first and second half.

Example 5.2. $\overset{+}{3}\ \overset{-}{1}\ \overset{-}{2}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{2}\ \overset{+}{1}\ \overset{-}{3}$ is a 0-sketch.

We will now look at some sub-arrangements of the type C arrangement. For each such arrangement, we will define the moves that we can apply to the 0-sketches (which correspond to changing inequalities of hyperplanes not in the arrangement) and then choose a canonical representative from each equivalence class to obtain a bijection with the regions of the sub-arrangement.

5.1.1 Boolean arrangement

One of first examples one encounters when studying hyperplane arrangement is the boolean arrangement. It is the arrangement in $\mathbb{R}^n$ with hyperplanes:

$$x_i = 0$$

for all $i \in [n]$. It is fairly straightforward to see that the number of regions is 2^n. We will do this using the idea of moves on 0-sketches. The hyperplanes missing from the type C arrangement in the boolean arrangement are:

$$x_i + x_j = 0$$
$$x_i - x_j = 0$$

for $1 \leq i < j \leq n$. Hence, the boolean moves, which we call B moves, are:

1. Swapping consecutive $\overset{+}{i}$ and $\overset{-}{j}$ as well as the corresponding negatives $\overset{+}{j}$ and $\overset{-}{i}$ where $i \neq j$ in $[n]$.
2. Swapping consecutive $\overset{+}{i}$ and $\overset{+}{j}$ as well as the corresponding negatives $\overset{-}{j}$ and $\overset{-}{i}$ where $i \neq j$ in $[n]$.

It can be shown that for any 0-sketch, we can use B moves to convert it to a 0-sketch where the order of absolute values in the first half is $1, 2, \ldots, n$. Also, the signs of the numbers in the first half do not change. Hence the number or boolean regions is the number of ways of assigning signs to the numbers $1, \ldots, n$ which is 2^n.

Example 5.3. We can convert $\overset{+}{3}\ \overset{-}{2}\ \overset{-}{1}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{1}\ \overset{+}{2}\ \overset{-}{3}$ to its canonical form as follows:

$$\overset{+}{3}\ \overset{-}{2}\ \overset{-}{1}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{1}\ \overset{+}{2}\ \overset{-}{3} \xrightarrow{B\ move} \overset{+}{3}\ \overset{-}{1}\ \overset{-}{2}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{2}\ \overset{+}{1}\ \overset{-}{3} \xrightarrow{B\ move} \overset{-}{1}\ \overset{+}{3}\ \overset{-}{2}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{2}\ \overset{-}{3}\ \overset{+}{1} \xrightarrow{B\ move} \overset{-}{1}\ \overset{-}{2}\ \overset{+}{3}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{-}{3}\ \overset{+}{2}\ \overset{+}{1}$$

5.1.2 Type D

The type D arrangement is the arrangement in $\mathbb{R}^n$ with the following hyperplanes:

$$x_i + x_j = 0$$
$$x_i - x_j = 0$$

for $1 \leq i < j \leq n$. So the hyperplanes missing from it are:

$$2x_i = 0$$

for all $i \in [n]$. Hence the type D moves, which we call D moves, are: Swapping $\overset{+}{i}$ and $\overset{-}{i}$ if they are consecutive for any $i \in [n]$. In an 0-sketch the only such pair is the last term of the first half and the first term of the second half. Hence D moves actually define an involution on the 0-sketches. Hence the number of regions of the type D arrangement is $2^{n-1}n!$. We could also choose a canonical sketch in each type D region to be the one where the last term of the first half is positive.

Example 5.4. $\overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{-}{2} \mid \overset{+}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4} \xrightarrow{D\ move} \overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4}$

5.1.3 Threshold arrangement

The threshold arrangement in $\mathbb{R}^n$ is the arrangement with hyperplanes:

$$x_i + x_j = 0$$

for all $1 \leq i < j \leq n$. So the hyperplanes missing from it are:

$$2x_i = 0$$
$$x_i - x_j = 0$$

for all $1 \leq i < j \leq n$. Hence the threshold moves, which we call T moves, are:

1. (D move) Swapping $\overset{+}{i}$ and $\overset{-}{i}$ if they are consecutive for any $i \in [n]$.
2. Swapping consecutive i and j as well as $\overset{-}{j}$ and $\overset{-}{i}$ where $i \neq j$ are of same sign.

Example 5.5. We can use a series of T moves on $\overset{+}{5}\,\overset{-}{4}\,\overset{-}{6}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{7}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{7}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{6}\,\overset{+}{4}\,\overset{-}{5}$ as follows:

$$\overset{+}{5}\,\overset{-}{4}\,\overset{-}{6}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{7}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{7}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{6}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{T\ move} \overset{+}{5}\,\overset{-}{4}\,\overset{-}{6}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{7}\,\overset{+}{3} \mid \overset{-}{3}\,\overset{-}{7}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{6}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{T\ move}$$
$$\overset{+}{5}\,\overset{-}{4}\,\overset{-}{6}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{7} \mid \overset{-}{7}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{6}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{T\ moves} \overset{+}{5}\,\overset{-}{1}\,\overset{-}{4}\,\overset{-}{6}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{7} \mid \overset{-}{7}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{6}\,\overset{+}{4}\,\overset{+}{1}\,\overset{-}{5}$$

We will call the set of numbers in a maximal string of completely positive or completely negative numbers in the first half of a sketch a *block*. The blocks of the initial sketch in Example 5.5 are: $\{5\}, \{1,4,6\}, \{2,7\}, \{3\}$ (these blocks appear in this order with the first one being positive). For any sketch, there is a T equivalent sketch (sketch that can be obtained using a series of T moves) for which the last block has more than 1 element. This is because, if the sketch has last block of size 1, swapping the n^{th} and $(n+1)^{th}$ term (D move), will result in a sketch where the last block has size greater than 1 (first step in Example 5.5). To obtain a canonical sketch for each threshold region, we will need a small lemma.

Lemma 5.1. *Two T equivalent sketches that have their last block of size greater than 1 have the same blocks which appear in the same order with the same signs.*

Proof. Looking at what the T moves do to the sequence of signs (above the numbers), we can see that they at most swap the n^{th} and $(n+1)^{th}$ sign. Hence, if we require the last blocks to have size greater than 1, both the sketches have the same number of blocks and the number of elements in the corresponding blocks are the same. A move of the second type can only reorder elements in the same block of a sketch. A move of the first kind changes the sign of the last element of the first half. So if there are $k > 1$ elements in the last block of a T equivalent sketch, then the set of mod values of the last k elements remains the same in all T equivalent sketches. Hence the elements of each block of T equivalent sketches with last block of size greater than 1 is also the same. □

Using the above lemma, we can see that for any sketch there is a unique T equivalent sketch where the size of the last block is greater than 1 and the elements of each block are ascending order of mod value. The last sketch in Example 5.5 is the unique such sketch in its equivalence class. To count the number of threshold regions, we count the number of such sketches, which is the same as the number of ways to form an ordered partition of $[n]$ with last block of size greater than 1 and then choosing whether the first block should be positive or negative:

$$\sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n,\ a_k\neq 1}} 2 \times \frac{n!}{a_1!\dots a_k!} = 2 \times (a(n) - n \times a(n-1)).$$

Here $a(n)$ is the n^{th} ordered Bell number.

5.2 Type C Catalan

The type C Catalan arrangement in $\mathbb{R}^n$ is the arrangement with hyperplanes:

$$\begin{aligned} 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. In this case, instead of looking at this arrangement directly, we will study the arrangement obtained by performing the translation $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$. The translated arrangement has hyperplanes:

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. This arrangement can be rewritten as:

$$\begin{gathered} x_i + 1 = -x_i - 1, \quad x_i + 1 = -x_i, \quad x_i = -x_i \\ x_i + 1 = -x_j - 1, \quad x_i + 1 = -x_j, \quad x_i = -x_j \\ x_i + 1 = x_j, \quad x_i = x_j, \quad x_i = x_j + 1 \end{gathered}$$

for all $1 \leq i < j \leq n$. So the regions of this arrangement are given by valid total orders on:

$$\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\}.$$

Such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in \{0,1\}$. Let $C^{(1)}(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [1,n],\ s \in \{0,1\}\} \cup \{\alpha_i^{(s)} \mid i \in [-n,-1],\ s \in \{-1,0\}\}.$$

Considering $-x_i$ as x_{-i}, $\alpha_i^{(s)}$ represents $x_i + s$ for any $\alpha_i^{(s)} \in C^{(1)}(n)$. For any $\alpha_i^{(s)} \in C^{(1)}(n)$, $\overline{\alpha_i^{(s)}}$ represents $\alpha_{-i}^{(-s)}$ and is called the conjugate of $\alpha_s^{(i)}$.

Example 5.6. The region

$$-x_3 - 1 < -x_3 < x_1 < -x_2 - 1 < x_1 + 1 < x_2 < -x_2 < -x_1 - 1 < x_2 + 1 < -x_1 < x_3 < x_3 + 1$$

is represented as: $\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)}\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_3^{(0)}\ \alpha_3^{(1)}$.

Definition 5.1 (Symmetric 1-sketch)**.** A word in the letters $C^{(1)}(n)$ which corresponds to a valid total order on $\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\}$ is called a symmetric 1-sketch. Hence symmetric 1-sketches correspond to regions of the type C Catalan arrangement.

Proposition 5.1. *A word in the letters $C^{(1)}(n)$ is a symmetric 1-sketch if and only if:*

1. *Each letter of $C^{(1)}(n)$ appears exactly once.*
2. *$\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)} \Rightarrow \alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.*
3. *$\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.*
4. *$\alpha_i^{(s)}$ appears before $\alpha_j^{(t)} \Rightarrow \overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.*

Just as for the usual Catalan arrangement, to prove this proposition, we have to show that there is a point in $\mathbb{R}^n$ satisfying the inequalities given by such a sketch. This will be proved in greater generality in Subsection 5.2.1.

We will now prove some special properties that symmetric 1-sketches satisfy. A symmetric 1-sketch has $4n$ letters, so we call the word made by the first $2n$ letters its first half. Similarly we define its second half.

Lemma 5.2. *The second half of a symmetric sketch is completely specified by its first half. In fact, it is the 'mirror' of the first half.*

Proof. For any symmetric 1-sketch,

$$\alpha_s^{(i)} \in \text{First half} \Leftrightarrow \overline{\alpha_s^{(i)}} \in \text{Second half}.$$

This property can be proved as follows: If there is some letter and its conjugate in the first half of a symmetric 1-sketch, there is some pair of conjugates in the second half as well (this is because conjugate pairs partition $C^{(1)}(n)$ into $2n$ pairs). But this would contradict property 4 of a symmetric 1-sketch in Proposition 5.1.

So the set of letters in the second half are the conjugates of the letters in the first half. The order in which they appear is forced by property 4, that is, the conjugates appear in the opposite order as the corresponding letters in the first half. So if the first half of a symmetric 1-sketch is $a_1 \ldots a_{2n}$ for some $a_i \in C^{(1)}(n)$ for all $i \in [2n]$, the sketch is:

$$a_1 \quad a_2 \quad \ldots \quad a_{2n} \quad \overline{a_{2n}} \quad \ldots \quad \overline{a_2} \quad \overline{a_1}.$$

□

We draw a vertical line between the $2n^{th}$ and $(2n+1)^{th}$ letter in a symmetric 1-sketch to indicate both the mirroring and the change in sign (note that if the $2n^{th}$ letter is $\alpha_i^{(s)}$, we have $x_i + s < 0 < -x_i - s$ in the corresponding region).

Example 5.7. $\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)} \mid \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$.

For a symmetric 1-sketch, an α-letter is a letter of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-1)}$ where $i \in [n]$. The other letters are called β-letters. The 'corresponding' α-letter (respectively β-letter) to a β-letter (respectively α-letter) is the one with the same subscript. Hence an α-letter always appears before its corresponding β-letter by property 3 in Proposition 5.1. The order in which the subscripts of the α-letters appear is the same as the order in which the subscripts of the α-letters appear by property 2 of Proposition 5.1. The proof of the following lemma is very similar to the proof of the previous lemma.

Lemma 5.3. *The order in which the subscripts of the α-letters in a symmetric 1-sketch appear is of the form:*

$$i_1 \quad i_2 \quad \ldots \quad i_n \quad -i_n \quad \ldots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$.

Using Lemmas 5.2 and 5.3, we only need to specify:

1. The α, β-word corresponding to the first half.
2. The signed permutation given by the first n α-letters.

to specify the sketch. The α, β-word corresponding to the first half is a word of length $2n$ in the letters $\{\alpha, \beta\}$ such that the i^{th} letter is an α if and only if the i^{th} letter of the symmetric 1-sketch is an α-letter. There is at most one sketch corresponding to a pair of an α, β-word and a signed permutation.

Example 5.8. For $\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)} \mid \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$,

1. α, β-word: $\alpha\ \beta\ \alpha\ \alpha\ \beta\ \alpha$.
2. Signed permutation: $-3 \quad 1 \quad -2$.

If we are given the α, β-word and signed permutation above, the unique sketch corresponding to it is the one given above. This is because the signed permutation tells us, by Lemma 5.3, that the order in which the subscripts of the α-letters (and hence β-letters) appears is: $-3 \quad 1 \quad -2 \quad 2 \quad -1 \quad 3$. So, using the α, β-word, we can construct the first half and, by Lemma 5.2, the entire sketch.

The next proposition characterizes the pairs of α, β-words and signed permutations that correspond to symmetric 1-sketches.

Proposition 5.2. *A pair of*

1. *An α, β-word of length $2n$ such that any prefix of the word has at least as many αs as βs.*
2. *Any signed permutation.*

corresponds to a symmetric 1-sketch and all symmetric 1-sketches correspond to such pairs.

Proof. By property 3 of symmetric 1-sketches, any α, β-word corresponding to the first half of a sketch should have at least as many αs as βs in any prefix.

We now prove that given such a pair, there is a symmetric 1-sketch corresponding to it. If the given α, β-word is $l_1\ l_2\ \ldots\ l_{2n}$ and the given signed permutation is $i_1\ i_2\ \ldots\ i_n$, we construct the symmetric 1-sketch as follows:

1. Extend the α, β-word to one of length $4n$ as:

$$l_1 \quad l_2 \quad \ldots \quad l_{2n} \quad \overline{l_{2n}} \quad \ldots \quad \overline{l_2} \quad \overline{l_1}.$$

   where $\overline{l_i} = \alpha \Leftrightarrow l_i = \beta$.

2. Extend the signed permutation to a sequence of length $2n$ as:

$$i_1 \quad i_2 \quad \ldots \quad i_n \quad -i_n \quad \ldots \quad -i_2 \quad -i_1.$$

3. Label the subscripts of the α-letters of the extended α, β-word in the order given by the extended signed permutation and similarly the β-letters.

If we show that the word constructed it a symmetric 1-sketch, it is clear that it will correspond to the given α, β-word and signed permutation. We have to check that the constructed word satisfies properties 1 to 4 of Proposition 5.1. The way the word was constructed, we see that it is of the form

$$a_1 \quad a_2 \quad \ldots \quad a_{2n} \quad \overline{a_{2n}} \quad \ldots \quad \overline{a_2} \quad \overline{a_1}.$$

for some $a_i \in C^{(1)}(n)$ for all $i \in [n]$. Since the conjugate of the i^{th} α is the $(2n-i+1)^{th}$ β and vice-versa, the first half of the word cannot have a pair of conjugates. Hence the word has all letters of $C^{(1)}(n)$. This gives that both property 1 and 4 hold. Property 2 is taken care of since we labeled the α and β-letters in the same order.

To show that property 3 holds, it is sufficient to show that any prefix of the word has at least as many αs as βs. This is already true for the first half. To show that this is true for the entire word, we look at α as $+1$ and β as -1. Hence the condition is that any prefix has a non-negative sum. Since any prefix (of size greater than $2n$) is of the form:

$$a_1 \quad a_2 \quad \ldots \quad a_{2n} \quad \overline{a_{2n}} \quad \ldots \quad \overline{a_k}$$

for some $1 \leq k \leq 2n$, the sum is 0 if $k = 1$ and $a_1 + \cdots + a_{k-1} \geq 0$ if $k > 1$. So property 3 holds as well and hence the constructed word is a symmetric 1-sketch. □

We can use this description to count the symmetric 1-sketches (the case of 1-sketches is simpler than that of m-sketches which will be defined later).

Lemma 5.4. *The number of α, β-words of length $2n$ having at least as many αs and βs in any prefix is $\binom{2n}{n}$.*

Proof. We will consider α as $+1$ and β as -1. So we have to show that there are $\binom{2n}{n}$ sequences consisting of $+1$s and -1s such that the sum of the first k terms is non-negative for all $k \in [2n]$.

We will use induction on n, the case of $n = 1$ being trivial. Given a sequence of length $2n + 2$ with all partial sums non-negative, removing the last two terms gives a sequence of length $2n$ with the same property. So, we count the number of ways of adding two terms to the end of a sequence of length $2n$ having all partial sums non-negative to obtain one of length $2n + 2$ with the same property.

If the sequence of length $2n$ has exactly n $+1$s, then it has exactly n -1s and hence has sum zero. So we cannot add the term -1 immediately after this sequence. We can either add $+1$ then -1 or two $+1$s. On the other hand, if the sequence of length $2n$ has more $+1$s than -1s, it has sum greater than zero. Since the length of the sequence is even, this sum cannot be 1. Hence we can add any two terms to the end of this sequence.

Since the number of sequences with n $+1$s and n -1s with all partial sums non-negative is the n^{th} Catalan number, $\frac{1}{n+1}\binom{2n}{n}$ (refer [22]), we get, by our induction hypothesis, that there are

$$2 \times \frac{1}{n+1}\binom{2n}{n} + 4 \times \left(\binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n}\right) = \binom{2n+2}{n+1}$$

sequences of length $2n + 2$ having all partial sums non-negative. □

Since there are $2^n n!$ signed permutations, the total number of symmetric 1-sketches and hence regions of the type C Catalan arrangement is

$$2^n n!\binom{2n}{n}.$$

5.2.1 Extended Catalan

Just as how the m-Catalan arrangement was defined as a deformation of the braid arrangement, the type C m-Catalan arrangement for any $m \geq 1$ is the arrangement in $\mathbb{R}^n$ with hyperplanes:

$$\begin{aligned} 2X_i &= -m, \dots, 0, \dots, m \\ X_i + X_j &= -m, \dots, 0, \dots, m \\ X_i - X_j &= -m, \dots, 0, \dots, m \end{aligned}$$

for all $1 \leq i < j \leq n$. Instead of looking at this arrangement directly, we will study the arrangement obtained by performing the translation $X_i = x_i + \frac{m}{2}$ for all $i \in [n]$. The translated arrangement has hyperplanes:

$$\begin{aligned} 2x_i &= -2m, \dots, 0 \\ x_i + x_j &= -2m, \dots, 0 \\ x_i - x_j &= -m, \dots, 0, \dots, m \end{aligned}$$

for all $1 \leq i < j \leq n$. The regions of this arrangement are given by valid total orders on:

$$\{x_i + s \mid i \in [n],\ s \in [0, m]\} \cup \{-x_i - s \mid i \in [n],\ s \in [0, m]\}.$$

Just as for the usual type C Catalan arrangement, such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in [0, m]$. Let $C^{(m)}(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [1, n],\ s \in [0, m]\} \cup \{\alpha_i^{(s)} \mid i \in [-n, -1],\ s \in [-m, 0]\}.$$

For any $\alpha_i^{(s)} \in C^{(m)}(n)$, $\overline{\alpha_i^{(s)}}$ represents $\alpha_{-i}^{(-s)}$ and is called the conjugate of $\alpha_s^{(i)}$. Letters of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-m)}$ for any $i \in [n]$ are called α-letters. The others are called β-letters.

Definition 5.2 (Symmetric m-sketch)**.** A word in the letters $C^{(m)}(n)$ which corresponds to a valid total order on $\{x_i + s \mid i \in [n],\ s \in [0, m]\} \cup \{-x_i - s \mid i \in [n],\ s \in [0, m]\}$ is called a symmetric m-sketch. Hence symmetric m-sketches correspond to regions of the type C m-Catalan arrangement.

Proposition 5.3. *A word in the letters $C^{(m)}(n)$ is a symmetric m-sketch if and only if:*

1. *Each letter of $C^{(m)}(n)$ appears exactly once.*
2. *$\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)} \Rightarrow \alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.*
3. *$\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.*
4. *$\alpha_i^{(s)}$ appears before $\alpha_j^{(t)} \Rightarrow \overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.*

Proof. We will prove this proposition by showing that there is a point in $\mathbb{R}^n$ satisfying the inequalities given by a word satisfying the above properties. The idea of the proof is same as that of Lemma 5.2 of [2]. Let w be a word in the letters $C^{(m)}(n)$ satisfying the properties given above. Using the same method as Lemma 5.2, we can prove that w is of the form

$$w_1 \quad w_2 \quad \ldots \quad w_{(m+1)n} \quad \overline{w_{(m+1)n}} \quad \ldots \quad \overline{w_2} \quad \overline{w_1}$$

for some $w_1, \ldots, w_{(m+1)n} \in C^{(m)}(n)$ none of which is a conjugate of the other. The idea is to construct a point satisfying the inequality defined by

$$w_k \quad w_{k+1} \quad \ldots \quad w_{(m+1)n} \quad \overline{w_{(m+1)n}} \quad \ldots \quad \overline{w_{k+1}} \quad \overline{w_k}$$

from a point satisfying the inequalities defined by

$$w_{k+1} \quad \ldots w_{(m+1)n} \quad \overline{w_{(m+1)n}} \quad \ldots \quad \overline{w_{k+1}}$$

for all $k \in [(m+1)n-1]$. By the way the inequalities are associated to the words, this amounts to choosing points $a_k < a_{k+1} < \cdots < a_{(m+1)n} < 0$ on the real line corresponding to $w_k, \ldots, w_{(m+1)n}$ respectively such that the distance between any pair corresponding to $\alpha_i^{(s-1)}$ and $\alpha_i^{(s)}$ are at a distance 1 apart given points $a'_{k+1} < \cdots < a'_{(m+1)n} < 0$ that satisfy the same property. The case $k = (m+1)n$ being satisfied by choosing any negative number.

First, suppose w_k is a letter of the form $\alpha_i^{(m)}$ or $\alpha_{-i}^{(0)}$. This means that there is no letter after it in the first half with the same subscript and hence choosing any $a_k < a'_{k+1}$ and taking $a_i = a'_i$ for all $i \in [k+1, (m+1)n]$, we get our required point. Next, suppose there is some letter after w_k in the first half which has the same subscript as it. Say w_l where $l \in [k+1, (m+1)n]$ is the first such letter, then by property 3 in the above list that w satisfies, we must have $a_k = a_l - 1$. We choose $a_k = a'_l - 1$ and $a_i = a'_i$ for all $i \in [l, (m+1)n]$. Let $j \in [k+1, l-1]$ be the smallest integer such that w_j has a letter after it in the first half with the same subscript. Take $a_i = a'_i$ for all $i \in [j, l-1]$ as well. Since $a_j - 1$ and $a_k - 1 = a_l$ are the same as $a'_j - 1$ and a'_l, by property 2 and our inductive hypothesis, we have that $a_j - 1 < a_k - 1$ and hence $a_j < a_k$. For the letters w_i for $i \in [k+1, j-1]$, since they do not have any letter with same subscript after them in the first half, we are free to choose any $a_{k+1} < \cdots < a_{j-1}$ between a_k and a_j. □

Similar to Lemma 5.3, it can be shown that the order in which the subscripts of the α-letters appear in a symmetric m-sketch is of the form

$$i_1 \quad i_2 \quad \ldots \quad i_n \quad -i_n \quad \ldots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$. Just as in the case of symmetric 1-sketches, we associate an α, β-word and signed permutation to a symmetric m-sketch which completely determines it.

Example 5.9. To the symmetric 2-sketch: $\alpha_2^{(0)}\alpha_{-1}^{(-2)}\alpha_2^{(1)}\alpha_{-1}^{(-1)}\alpha_1^{(0)}\alpha_{-2}^{(-2)} \mid \alpha_2^{(2)}\alpha_{-1}^{(0)}\alpha_1^{(1)}\alpha_{-2}^{(-1)}\alpha_1^{(2)}\alpha_{-2}^{(0)}$ we associate:

1. α, β-word: $\alpha\alpha\beta\beta\alpha\alpha$.

2. Signed permutation: $2 \ -1$.

The set of α, β-words associated to symmetric m-sketches for $m > 1$ does not seem to have a simple characterization like those for symmetric 1-sketches (see Proposition 5.2). However, looking at symmetric m-sketches as certain labeled non-nesting partitions as done in [2], we see that such objects have already been counted bijectively (refer [7]).

In [2], C. A. Athanasiadis, obtains a bijection with several classes of non-nesting partitions and regions of certain arrangements. We will mention the one for the translated type C m-Catalan arrangement, which gives a bijection between the α, β-words associated to symmetric m-sketches and certain non-nesting partitions.

Definition 5.3 (Symmetric non-nesting partition)**.** A symmetric m-non-nesting partition is a partition of $[-(m+1)n, (m+1)n]$ such that

1. Each block is of size $(m+1)$.
2. If B is a block, so is $-B$.
3. If a, b are in some block B, $a < b$ and there is no number $a < c < b$ such that $c \in B$, then if $a < c < d < b$, c and d are not in the same block.

Consider the numbers $[-(m+1)n, (m+1)n]$ arranged as:

$$-(m+1)n \quad \ldots \quad -2 \quad -1 \quad 1 \quad 2 \quad \ldots (m+1)n$$

If the consecutive elements in each block of a symmetric m-non-nesting partition are joined by arcs, the diagram we get is symmetric and without nesting. It can also be seen that there are exactly n pairs of blocks of the form $\{B, -B\}$ with no block containing both a number and its negative. Also, the first n blocks, with blocks being read in order of the smallest element in it, do not have a pair of the form $\{B, -B\}$. Hence we can label the first n blocks with a signed permutation and label $-B$ the negative of the label of B for all blocks B. Such objects will be called labeled symmetric m-non-nesting partitions.

We can obtain a labeled symmetric m-non-nesting partition from a symmetric m-sketch by writing $-(m+1)n$ for the first letter and so on till $(m+1)n$ for the last letter and joining the letters $\alpha_i^{(0)}, \alpha_i^{(1)}, \ldots, \alpha_i^{(m)}$ and similarly $\alpha_{-i}^{(-m)}, \ldots, \alpha_{-i}^{(0)}$ with arcs and labeling each such chain with the subscript of the letters being joined. It can be shown that this construction is a bijection between symmetric m-sketches and labeled symmetric m-non-nesting partitions. Hence the α, β-words associated with symmetric m-sketches are in bijection with symmetric m-non-nesting partitions.

Example 5.10. To the symmetric 2-sketch: $\alpha_2^{(0)}\alpha_{-1}^{(-2)}\alpha_2^{(1)}\alpha_{-1}^{(-1)}\alpha_1^{(0)}\alpha_{-2}^{(-2)} \mid \alpha_2^{(2)}\alpha_{-1}^{(0)}\alpha_1^{(1)}\alpha_{-2}^{(-1)}\alpha_1^{(2)}\alpha_{-2}^{(0)}$ we associate the labeled 2-non-nesting partition of Figure 5.1.

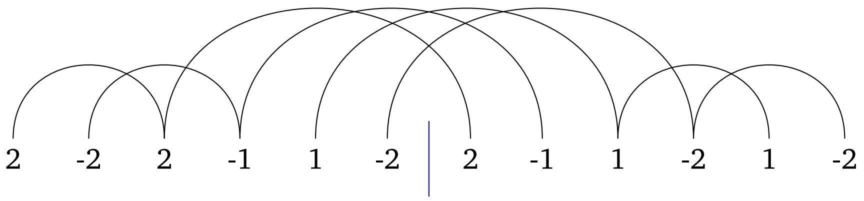


Figure 5.1: A labeled 2-non-nesting partition

The number of various classes of non-nesting partitions have been counted bijectively. In terms of [7] or [2], the symmetric m-non-nesting partitions defined above are called type C partitions of size $(m+1)n$ of type $(m+1, \ldots, m+1)$ where this is an n-tuple representing the size of the (nonzero) block pairs $\{B, -B\}$. The number of such partitions is

$$\binom{(m+1)n}{n}$$

and hence the number of symmetric m-sketches, which is the number of type C m-Catalan regions is

$$2^n n! \binom{(m+1)n}{n}.$$

5.3 Catalan arrangement in other root systems

We will now use 'moves', as in [5], to count the regions of Catalan arrangements in other root systems.

5.3.1 Type D Catalan

The type D Catalan arrangement is the arrangement in $\mathbb{R}^n$ with the hyperplanes:

$$X_i + X_j = -1, 0, 1$$
$$X_i - X_j = -1, 0, 1$$

for $1 \leq i < j \leq n$. Translating this arrangement by putting $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$, we get:

$$x_i + x_j = -2, -1, 0$$
$$x_i - x_j = -1, 0, 1$$

for $1 \leq i < j \leq n$. Figure 5.2 shows how this arrangement is a sub-arrangement of the type C Catalan arrangement in $\mathbb{R}^2$. It also shows how the regions of the type D Catalan arrangement partition the regions of the type C Catalan arrangement.

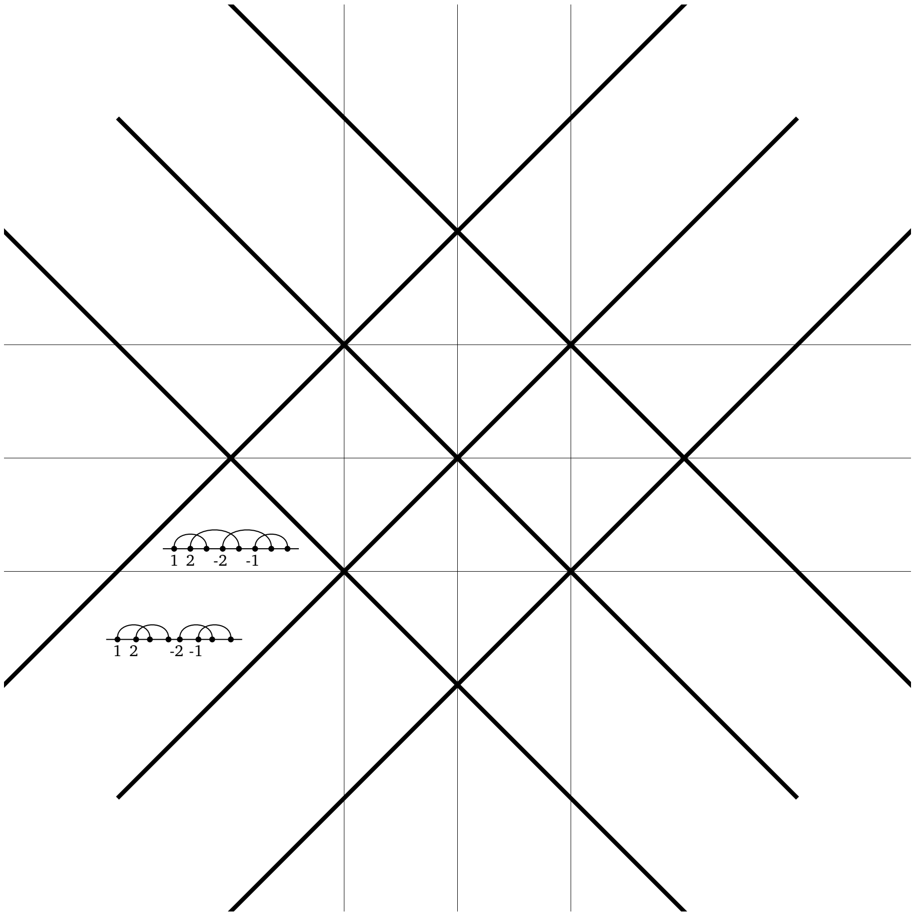


Figure 5.2: Type C Catalan arrangement in $\mathbb{R}^2$. Two type C Catalan regions are labeled with their symmetric labeled non-nesting partition. Bold lines are the type D Catalan hyperplanes.

We use the idea of 'moves' to count the number of type D Catalan regions. The hyperplanes missing from the translated type C Catalan arrangement in the translated type D Catalan arrangement are:

$$2x_i = -2, -1, 0$$

for all $i \in [n]$. So, the type D Catalan moves, which we call $\mathrm{D}^{(1)}$ moves, are:

1. Corresponding to $2x_i = -2$, $2x_i = 0$: Swapping the $2n^{th}$ and $(2n+1)^{th}$ letter.
2. Corresponding to $2x_i = -1$: Swapping the n^{th} and $(n+1)^{th}$ α if they are consecutive (along with the n^{th} and $(n+1)^{th}$ β).

The first move covers the inequalities of the type $2x_i = -2$ or $2x_i = 0$ (which is the same as $x_i + 1 = -x_i - 1$ or $x_i = -x_i$) since the only conjugates that are consecutive, by Lemma 5.2, are the $2n^{th}$ and $(2n+1)^{th}$ letter.

The second move covers the inequalities of the type $2x_i = -1$ (which is the same as $x_i = -x_i - 1$ and $x_i + 1 = -x_i$) since the only way $\alpha_i^{(0)}$ and $\alpha_{-i}^{(-1)}$ as well as $\alpha_i^{(1)}$ and $\alpha_{-i}^{(0)}$ can be consecutive is, by Lemma 5.3, when the n^{th} and $(n+1)^{th}$ α-letters are consecutive. Also, by Lemma 5.2, the n^{th} and $(n+1)^{th}$ α-letters are consecutive if and only if the n^{th} and $(n+1)^{th}$ β-letters are consecutive.

Example 5.11. A series of $\mathrm{D}^{(1)}$ moves applied to a symmetric 1-sketch is given below:

$$\begin{aligned}
&\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathrm{D}^{(1)}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathrm{D}^{(1)}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathrm{D}^{(1)}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)}
\end{aligned}$$

To count the number of regions of the type D Catalan arrangement, we have to count the number of equivalence classes of symmetric 1-sketches where two sketches are equivalent if one can be obtained from the other via a series of $\mathrm{D}^{(1)}$ moves. In Figure 5.2, it can be seen that the two labeled type C Catalan regions are adjacent and lie in the same type D Catalan region. They are related by swapping of the fourth and fifth letters of their sketches, which is a $\mathrm{D}^{(1)}$ move.

The fact about these moves that will help with the count is that a series of $\mathrm{D}^{(1)}$ moves cannot change the sketch too much. Hence we can list the sketches that are $\mathrm{D}^{(1)}$ equivalent to a given sketch.

First, consider the case when the n^{th} α-letter of the symmetric 1-sketch is not in the $(2n-1)^{th}$ position. In this case, the n^{th} α-letter is far enough from the $2n^{th}$ letter that a $\mathrm{D}^{(1)}$ move of the first kind (swapping the $2n^{th}$ and $(2n+1)^{th}$ letter) will not affect the letter after the n^{th} α-letter. Hence it does not change whether the n^{th} and $(n+1)^{th}$ α-letters are consecutive.

The number of sketches equivalent to a sketch when the n^{th} α-letter is not in the $(2n-1)^{th}$ position and:

1. n^{th} and $(n+1)^{th}$ α-letter are consecutive is 4.

$$\begin{gathered}
\ldots\alpha_{-i}^{(-1)}\alpha_{i}^{(0)}\ldots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\ldots\alpha_{-i}^{(0)}\alpha_{i}^{(1)}\ldots\\
\ldots\alpha_{-i}^{(-1)}\alpha_{i}^{(0)}\ldots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\ldots\alpha_{-i}^{(0)}\alpha_{i}^{(1)}\ldots\\
\ldots\alpha_{i}^{(0)}\alpha_{-i}^{(-1)}\ldots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\ldots\alpha_{i}^{(1)}\alpha_{-i}^{(0)}\ldots\\
\ldots\alpha_{i}^{(0)}\alpha_{-i}^{(-1)}\ldots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\ldots\alpha_{i}^{(1)}\alpha_{-i}^{(0)}\ldots
\end{gathered}$$

2. n^{th} and $(n+1)^{th}$ α-letter are not consecutive is 2.

$$\ldots\alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)}\ldots \qquad \ldots\alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)}\ldots$$

Notice also that the equivalent sketches also satisfy the same properties (n^{th} α-letter not being in the $(2n-1)^{th}$ position and whether n^{th} and $(n+1)^{th}$ α-letters are consecutive).

In case the n^{th} α-letter is in the $(2n-1)^{th}$ position of the symmetric 1-sketch, it can be checked that it has exactly 4 equivalent sketches all of which also have the n^{th} α-letter in the

$(2n-1)^{th}$ position.

$$\ldots \alpha_i^{(0)} \alpha_i^{(1)} \mid \alpha_{-i}^{(-1)} \alpha_{-i}^{(0)} \ldots$$
$$\ldots\ \alpha_i^{(0)} \alpha_{-i}^{(-1)} \mid \alpha_i^{(1)} \alpha_{-i}^{(0)} \ldots$$
$$\ldots \alpha_{-i}^{(-1)} \alpha_i^{(0)} \mid \alpha_{-i}^{(0)} \alpha_i^{(1)} \ldots$$
$$\ldots\ \alpha_{-i}^{(-1)} \alpha_{-i}^{(0)} \mid \alpha_i^{(0)} \alpha_i^{(1)} \ldots$$

Figure 5.2 shows that in $\mathbb{R}^2$, each type D Catalan region contains exactly 4 or exactly 2 type C Catalan regions, as expected from the above observations.

Notice that the number of sketches equivalent to a given sketch only depends on its α, β-word (see Proposition 5.2). So, we need to count the number of α, β-words of length $2n$ with any prefix having at least as many αs as βs such that:

1. The n^{th} α-letter is not in the $(2n-1)^{th}$ position and
   (a) The letter after the n^{th} α-letter is an α.
   (b) The letter after the n^{th} α-letter is a β.
2. The n^{th} α-letter is in the $(2n-1)^{th}$ position.

We first count the second type of α, β-words. If the n^{th} α-letter is in the $(2n-1)^{th}$ position, the first $(2n-2)$ letters have $(n-1)$ αs and $(n-1)$ βs. The $(2n-1)^{th}$ and $2n^{th}$ letters are α and β or α and α respectively. So the total number of such α, β-words is

$$2 \times \frac{1}{n}\binom{2n-2}{n-1}$$

since the number of sequences of length $(2n-2)$ having $(n-1)$ αs and having at least as many αs as βs in any prefix is the $(n-1)^{th}$ Catalan number (see [22]).

The number of both the types 1(a) and 1(b) of α, β-words mentioned above are the same. This is because, on the set of α, β-word of length $2n$ with any prefix having at least as many αs as βs, changing the letter after the n^{th} α-letter is an involution. Since we have the number of words that do not have the n^{th} α-letter in the $(2n-1)^{th}$ position and the total number of words, the number of words of type 1(a) and 1(b) are both equal to:

$$\frac{1}{2} \times \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right].$$

Combining the observations made above, we get that the number of type D Catalan regions is

$$2^n n! \times \left(\frac{1}{4} \times \left[\frac{2}{n}\binom{2n-2}{n-1} + \frac{1}{2} \times \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right]\right] + \frac{1}{2} \times \left[\frac{1}{2} \times \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right]\right]\right)$$

which simplifies to

$$2^{n-1} \times \frac{(2n-2)!}{(n-1)!} \times (3n-2).$$

5.3.2 Type B Catalan

The type B Catalan arrangement in $\mathbb{R}^n$ has the hyperplanes:

$$\begin{aligned} X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. Translating this arrangement by putting $X_i = x_i + \frac{1}{2}$, we get the arrangement:

$$\begin{aligned} x_i &= -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. While it is possible to consider this arrangement as a sub-arrangement of the translated type C 2-Catalan arrangement (see Subsection 5.2.1), this would add too many extra hyperplanes. Also, we do not have a simple characterization of symmetric 2-sketches, as we do for symmetric 1-sketches (see Proposition 5.2).

We instead consider this translated type B Catalan arrangement as a sub-arrangement of the arrangement in $\mathbb{R}^n$ which has hyperplanes:

$$\begin{aligned} x_i &= -\frac{5}{2}, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned} \tag{5.1}$$

for all $1 \leq i < j \leq n$. Rewriting the hyperplanes of the arrangement as

$$\begin{aligned} &x_i + 1 = -\frac{3}{2},\ x_i = -\frac{3}{2},\ x_i + 1 = -x_i - 1,\ x_i = -\frac{1}{2},\ x_i = -x_i,\ x_i = \frac{1}{2},\ x_i = \frac{3}{2} \\ &x_i + 1 = -x_j - 1,\ x_i + 1 = -x_j,\ x_i = -x_j \\ &x_i + 1 = x_j,\ x_i = x_j,\ x_i = x_j + 1 \end{aligned}$$

for all $1 \leq i < j \leq n$, we can see that a region of this arrangement is given by a valid total order on:

$$\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}.$$

Now we define sketches that represent such orders. We will represent $x_i + s$ as $\alpha_i^{(s)}$ and $-x_i - s$ as $\alpha_{-i}^{(-s)}$ for any $i \in [n]$ and $s \in \{0,1\}$. $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ will be represented as $\alpha_-^{(-1.5)}$, $\alpha_-^{(-0.5)}$, $\alpha_+^{(0.5)}$, $\alpha_+^{(1.5)}$ respectively.

Example 5.12. The region

$$-\frac{3}{2} < x_2 < -x_1 - 1 < -\frac{1}{2} < x_1 < x_2 + 1 < -x_2 - 1 < -x_1 < \frac{1}{2} < x_1 + 1 < -x_2 < \frac{3}{2}$$

is represented as $\alpha_-^{(-1.5)}\ \alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_-^{(-0.5)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha_+^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}\ \alpha_+^{(1.5)}$.

Call the words in the letters

$$B^{(1)}(n) = \{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\} \cup \{\alpha_-^{(-1.5)}, \alpha_-^{(-0.5)}, \alpha_+^{(0.5)}, \alpha_+^{(1.5)}\}$$

that correspond to regions 'valid sketches'. Denote by $\overline{\alpha_x^{(s)}}$ the letter $\alpha_{-x}^{(-s)}$ for any $\alpha_x^{(s)} \in B^{(1)}(n)$. We have the following characterization of valid sketches:

Proposition 5.4. *A word in the letters $B^{(1)}(n)$ is a valid sketch if and only if:*

1. *Each letter of $B^{(1)}(n)$ appears exactly once.*
2. *$\alpha_x^{(s-1)}$ appears before $\alpha_y^{(t-1)} \Rightarrow \alpha_x^{(s)}$ appears before $\alpha_y^{(t)}$.*
3. *$\alpha_x^{(s-1)}$ appears before $\alpha_x^{(s)}$.*
4. *$\alpha_x^{(s)}$ appears before $\alpha_y^{(t)} \Rightarrow \overline{\alpha_y^{(t)}}$ appears before $\overline{\alpha_x^{(s)}}$.*

Just as was done for the type C sketches, we associate a symmetric non-nesting diagram to each valid sketch and can inductively construct a point satisfying the inequality specified by a valid sketch (see 5.2.1). Also, just as for type C sketches, it can be shown that the valid sketches are symmetric about the center.

Example 5.13. To the valid sketch given below, we associate the arc diagram in Figure 5.3.

$$\alpha_-^{(-1.5)}\alpha_2^{(0)}\alpha_{-1}^{(-1)}\alpha_-^{(-0.5)}\alpha_1^{(0)}\alpha_2^{(1)} \mid \alpha_{-2}^{(-1)}\alpha_{-1}^{(0)}\alpha_+^{(0.5)}\alpha_1^{(1)}\alpha_{-2}^{(0)}\alpha_+^{(1.5)}$$

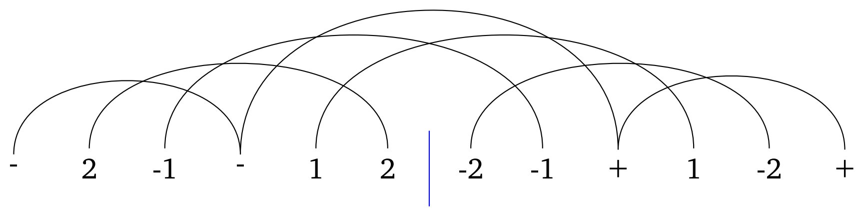


Figure 5.3: Arc diagram associated to $\alpha_-^{(-2)}\ \alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_-^{(-1)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha_+^{(1)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}\ \alpha_+^{(2)}$

To valid sketches, we can associate a pointed α, β-word of length $(2n+2)$ and a signed permutation as follows:

1. For the letters in the first half of the valid sketch of the form $\alpha_i^{(0)}$, $\alpha_{-i}^{(-1)}$ or $\alpha_-^{(-1.5)}$ for any $i \in [n]$, we write α and for the others we write β (α corresponds to 'openers' in the arc diagram and β to 'closers'). The β corresponding to $\alpha_-^{(0.5)}$ is pointed to.
2. The subscripts of the first n α letters other than $\alpha_-^{(-1.5)}$ gives us the signed permutation.

Example 5.14. To the valid sketch $\alpha_-^{(-1.5)}\ \alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_-^{(-0.5)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha_+^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}$ $\alpha_+^{(1.5)}$, we associate:

1. Pointed α, β-word: $\alpha\alpha\alpha\beta\alpha\beta$.
2. Signed permutation: $2\ -1$.

Just as was done for type C sketches, we can see that the method given above to get a signed permutation does actually give a signed permutation and that the pointed α, β-word satisfies the property that in any prefix, there are at least as many α-letters as β-letters. Also, just as in type C sketches, such a pair has at most one valid sketch associated to it. We now characterize the pointed α, β-words and signed permutations associated to valid sketches.

Proposition 5.5. *A pair of*

1. *A pointed α, β-word satisfying the property that in any prefix, there are at least as many α-letters as β-letters and that the number of α-letters before the pointed β is $(n+1)$.*
2. *Any signed permutation.*

corresponds to a valid sketch and all valid sketches correspond to such pairs.

Proof. Most of the proof is just the same as for the type C sketches. The main difference is pointing at the $\alpha_-^{(-0.5)}$ β-letter. The property we have to take care of is that there is no nesting in the arc joining $\alpha_-^{(0.5)}$ to $\alpha_+^{(0.5)}$. This is the same as finding a condition for an arc drawn from a β-letter in the first half to its mirror image in the second half to not cause any nesting.

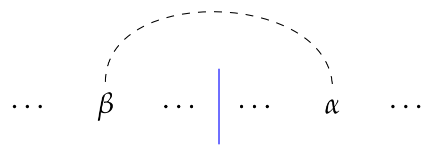


Figure 5.4: Arc from β to its mirror image.

Denote by $N_{\alpha,b}$ the number of α-letters before the β under consideration, $N_{\alpha,a}$ the number of α-letters in the first half after the β and similarly define $N_{\beta,b}$ and $N_{\beta,a}$. The condition that we do not want an arc inside the one joining the β to its mirror is given by:

$$N_{\alpha,b} \geq N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a} \tag{5.2}$$

This is because of the symmetry of the arc diagram and the fact that we want any β-letter between the pointed β and its mirror to have its corresponding α before the pointed β. Similarly, the condition that we do not want the arc joining the β to its mirror to be contained in any arc is given by:

$$N_{\alpha,b} \leq N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a} \tag{5.3}$$

This is because of the symmetry of the arc diagram and the fact that we want any α-letter before the pointed β to have its corresponding β before the mirror of the pointed β. Combining the conditions (5.2) and (5.3), we get:

$$N_{\alpha,b} = N_{\beta,b} + 1 + N_{\beta,a} + N_{\alpha,a}.$$

But this is saying that the number of α-letters before the pointed β should be equal to the number of remaining letters in the first half. Since the total number of letters in the first half is $(2n+2)$, we get that: The arc joining a β in the first half to its mirror does not cause nesting problems if and only if the number of α letters before it is $(n+1)$. $\square$

Now we go back to the translated type B arrangement. The hyperplanes missing from (5.1) are:

$$x_i = -\frac{5}{2}, -1, 0, \frac{3}{2}$$

for all $i \in [n]$. Hence the moves on valid sketches, which we call B moves, corresponding to changing one of these inequalities are:

1. Corresponding to $x_i = 0$, $x_i = -1$: Swapping to $(2n+2)^{th}$ and $(2n+3)^{th}$ letter if they are not $\alpha_-^{(0.5)}$ and $\alpha_+^{(0.5)}$.

2. Corresponding to $x_i = -\frac{5}{2}$, $x_i = \frac{3}{2}$: Swapping the pointed β, that is, $\alpha_-^{(-0.5)}$ and a β-letter immediately before or after it (and making the corresponding change in the second half).

We can see that such moves change the pointed α, β-word associated to a sketch by at most changing the last letter or changing which of the β-letters between the $(n+1)^{th}$ and $(n+2)^{th}$ α-letter (or just after the $(n+1)^{th}$ α if there are only $(n+1)$ α-letters) is pointed to. So if we force that the last letter of the sketch has to be a β-letter and that the β-letter immediately after the $(n+1)^{th}$ α-letter has to be pointed to, we get a canonical sketch in each equivalence class. We will call such sketches type B sketches.

Since there is no condition on the signed permutation, we will now count α, β-words associated to type B sketches. From Proposition 5.5, we can see that the α, β-words we need to count are those such that:

1. Length of the word is $(2n+2)$.

2. In any prefix, there are at least as many α-letters as β-letters.

3. The letter immediately after the $(n+1)^{th}$ α-letter is a β (pointed β).

4. The last letter is a β.

The number of words satisfying all properties but also has the $(n+1)^{th}$ α-letter in the $(2n+1)^{th}$ position is the n^{th} Catalan number. This is because the $2n$ letters before the $(n+1)^{th}$ α need to have exactly n αs and satisfy property 2. Hence such words can be constructed by adding α as the $(2n+1)^{th}$ and β as the $(2n+2)^{th}$ letter (to satisfy property 4) to such a ballot sequence of length $2n$ (refer [22]). So, the number of α, β-words satisfying all properties listed above and has the $(n+1)^{th}$ α-letter in the $(2n+1)^{th}$ position is:

$$\frac{1}{n+1}\binom{2n}{n}.$$

We know that the number of α, β-words satisfying properties 1 and 2 is $\binom{2n+2}{n+1}$ (from type C counting). Also, on the set of α, β-words satisfying properties 1 and 2, changing the letter immediately after the $(n+1)^{th}$ α-letter (α to β or β to α) is an involution. Changing the last letter of such a words is also an involution. If the $(n+1)^{th}$ α-letter for a sketch is not in the $(2n+1)^{th}$ position, its orbit under these two involutions has 4 sketches of which exactly one satisfies all properties listed above. So, the number of α, β-words satisfying all properties listed above and does not have the $(n+1)^{th}$ α-letter in the $(2n+1)^{th}$ position is:

$$\frac{1}{4} \times \left(\binom{2n+2}{n+1} - 2 \times \frac{1}{n+1}\binom{2n}{n} \right).$$

The $-2 \times \frac{1}{n+1}\binom{2n}{n}$ removes the α, β-words satisfying property 1 and 2 with the $(n+1)^{th}$ α-letter at the $(2n+1)^{th}$ position (ballot sequence of length $2n$ + $\alpha\alpha$ or ballot sequence of length $2n$ + $\alpha\beta$). So the total number of α, β-words satisfying the required properties is:

$$\frac{1}{n+1}\binom{2n}{n} + \frac{1}{4} \times \left(\binom{2n+2}{n+1} - 2 \times \frac{1}{n+1}\binom{2n}{n} \right) = \binom{2n}{n}.$$

Hence, the number of type B sketches, which is the number of regions of the type B Catalan arrangement in $\mathbb{R}^n$ is:

$$2^n n! \binom{2n}{n}.$$

5.3.3 Type BC Catalan

The type BC Catalan arrangement in $\mathbb{R}^n$ is the arrangement with hyperplanes:

$$\begin{aligned} X_i &= -1, 0, 1 \\ 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. Translating this arrangement by putting $X_i = x_i + \frac{1}{2}$, we get the arrangement:

$$\begin{aligned} x_i &= -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leq i < j \leq n$. Again we consider this arrangement as a sub-arrangement of (5.1). To define moves on valid sketches note that the hyperplanes missing from (5.1) are:

$$x_i = -\frac{5}{2}, \frac{3}{2}$$

for all $i \in [n]$. Hence the moves on valid sketches, which we call BC moves, corresponding to changing one of the inequalities corresponding to $x_i = -\frac{5}{2}$ or $x_i = \frac{3}{2}$ are: Swapping the pointed

β, that is, $\alpha_{-}^{(-0.5)}$ and a β-letter immediately before or after it (and making the corresponding change in the second half).

We can see that such moves change the pointed α, β-word associated to a sketch by at most changing which of the β-letters between the $(n+1)^{th}$ and $(n+2)^{th}$ α-letter (or just after the $(n+1)^{th}$ α if there are only $(n+1)$ α-letters) is pointed to. So if we force that the β-letter immediately after the $(n+1)^{th}$ α-letter has to be pointed to, we get a canonical sketch in each equivalence class. We will call such sketches type BC sketches.

So we have to count the number of α, β-words such that:

1. Length of the word is $(2n+2)$.

2. In any prefix, there are at least as many α-letters as β-letters.

3. The letter immediately after the $(n+1)^{th}$ α-letter is a β (pointed β).

Using the involution on the set of words satisfying properties 1 and 2 of changing the letter immediately after the $(n+1)^{th}$ α-letter and the fact that there are $\binom{2n+2}{n+1}$ words satisfying properties 1 and 2, we get that the number of words satisfying the required properties is:

$$\frac{1}{2} \times \binom{2n+2}{n+1}.$$

Hence, the number of type BC sketches, which is the number of regions of the type BC Catalan arrangement in $\mathbb{R}^n$ is:

$$2^{n-1} n! \binom{2n+2}{n+1}.$$

Chapter 6

Boxed threshold arrangement

In this chapter, we consider the hyperplane arrangement $\mathcal{BT}_n$ in $\mathbb{R}^n$ whose hyperplanes are $\{X_i + X_j = 1 \mid 1 \leq i < j \leq n\} \cup \{X_i = 0, 1 \mid 1 \leq i \leq n\}$. This arrangement has been studied via graphs in a series of papers by Joungmin Song ([18], [17], [19]). First we obtain the characteristic polynomial of the arrangement via the finite field method. We will then show how the method of sketches and moves makes counting of its regions simpler and also exhibit a bijection between the regions and certain colored threshold graphs.

6.1 The characteristic polynomial

First translate the hyperplanes in $\mathcal{BT}_n$ in order to obtain a combinatorially isomorphic arrangement with the same characteristic polynomial. Putting $X_i = x_i + \frac{1}{2}$ for every i we get:

$$\{x_i + x_j = 0 \mid 1 \leq i < j \leq n\} \cup \{x_i = -\frac{1}{2}, \frac{1}{2} \mid 1 \leq i \leq n\}.$$

We will stick to the notation $\mathcal{BT}_n$ to denote the above arrangement. Consider the following relationship between the characteristic polynomial of certain central arrangements and that of their "boxed" versions.

Proposition 6.1. *Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ that is a sub-arrangement of the type C arrangement, that is, a sub-arrangement of $\{x_i \pm x_j = 0 \mid 1 \leq i < j \leq n\} \cup \{x_i = 0 \mid i \in [n]\}$ and let $\mathcal{BA} = \mathcal{A} \cup \{x_i = -\frac{1}{2}, \frac{1}{2} \mid i \in [n]\}$. Then*

$$\chi_{\mathcal{BA}}(t) = \chi_{\mathcal{A}}(t-2).$$

Proof. The proof is by using the form of the finite field method given in Theorem 2.1. Let q be any large odd number. Set $D_q^n := \{(a_1, \ldots, a_n) \in \mathbb{Z}_q^n \mid a_i \neq \pm\frac{q-1}{2}\}$. Define a bijection $f : \mathbb{Z}_{q-2} \to \mathbb{Z}_q \setminus \{\frac{q-1}{2}, -\frac{q-1}{2}\}$ as

$$f(i) = i \quad \text{for } i \in [-\frac{q-3}{2}, \frac{q-3}{2}].$$

It is clear that for any $a, b \in \mathbb{Z}_{q-2}$

1. $a + b = 0$ if and only if $f(a) + f(b) = 0$.
2. $a - b = 0$ if and only if $f(a) - f(b) = 0$.
3. $a = 0$ if and only if $f(a) = 0$.

Using f, we can define a bijection $F : \mathbb{Z}_{q-2}^n \to D_q^n$ as

$$F(a_1, \dots, a_n) = (f(a_1), \dots, f(a_n)) \quad \text{for } (a_1, \dots, a_n) \in \mathbb{Z}_{q-2}^n.$$

By the properties of f, we can see that F induces a bijection between those tuples in $\mathbb{Z}_{q-2}^n$ that do not satisfy the defining equation of any hyperplane in $\mathcal{A}$ and those tuples in $\mathbb{Z}_q^n$ that do not satisfy the defining equation of any hyperplane in $\mathcal{BA}$. So, we get that for large odd numbers q,

$$\chi_{\mathcal{BA}}(q) = \chi_{\mathcal{A}}(q - 2).$$

Since $\chi_{\mathcal{BA}}$ and $\chi_{\mathcal{A}}$ are polynomials, we get the required result. □

Denote by $\mathcal{T}_n$ the threshold arrangement in $\mathbb{R}^n$, i.e., $\mathcal{T}_n := \{x_i + x_j = 0 \mid 1 \leq i < j \leq n\}$. The reason this is called the threshold arrangement is that its regions are in bijection with labeled threshold graphs on n vertices (see Section 6.3 for details). This is clearly a sub-arrangement of type C arrangement.

Corollary 6.1. *The characteristic polynomials of $\mathcal{BT}_n$ and $\mathcal{T}_n$ are related as follows:*

$$\chi_{\mathcal{BT}_n}(t) = \chi_{\mathcal{T}_n}(t - 2).$$

Consequently, the number of bounded regions of $\mathcal{BT}_n$ is equal to the number of regions of $\mathcal{T}_n$. Moreover, these bounded regions are contained in the cube (or a *box*) $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Next, we derive a closed form expression for $\chi_{\mathcal{T}_n}(t)$ using the finite field method.

Proposition 6.2. *The characteristic polynomial of the threshold arrangement $\mathcal{T}_n$ is*

$$\chi_{\mathcal{T}_n}(t) = \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{i=1}^{k} (t - (2i - 1)).$$

Here $S(n,k)$ are the Stirling numbers of the second kind.

Proof. Using the finite field method, we see that the characteristic polynomial of $\mathcal{T}_n$ satisfies, for large values of q,

$$\chi_{\mathcal{T}_n}(q) = |\{(a_1, \dots, a_n) \in \mathbb{Z}_q^n \mid a_i + a_j \neq 0 \text{ for all } 1 \leq i < j \leq n\}|.$$

This means that we need to count the functions $f : [n] \to \mathbb{Z}_q$ such that:

1. There is at most one $i \in [n]$ such that $f(i) = 0$.
2. f can take at most one value from each of the sets

$$\{1, -1\}, \{2, -2\}, \dots, \{\frac{q-1}{2}, -\frac{q-1}{2}\}.$$

We split the count into the two cases. If 0 is not attained by f, then all values must be from

$$\{1,-1\} \cup \{2,-2\} \cup \cdots \cup \{\frac{q-1}{2}, -\frac{q-1}{2}\}.$$

with at most one value attained in each set. So, there are

$$\binom{\frac{q-1}{2}}{k} \times 2^k \times k!S(n,k)$$

ways for f to attain values from exactly k of these sets. Since we have $\binom{\frac{q-1}{2}}{k} \times 2^k$ ways to choose the k sets and which element of each set f should attain and $k!S(n,k)$ ways to choose the images of the elements of $[n]$ after making this choice. So the total number of f such that 0 is not attained is:

$$\sum_{k=1}^{n} \binom{\frac{q-1}{2}}{k} \times 2^k \times k!S(n,k).$$

When 0 is attained, there are n ways to choose which element of $[n]$ gets mapped to 0 and using a similar logic for choosing the images of the other elements, we get that the total number of f where 0 is attained is:

$$n \times \sum_{k=1}^{n-1} \binom{\frac{q-1}{2}}{k} \times 2^k \times k!S(n-1,k).$$

So we get that for large q,

$$\begin{aligned} \chi_{\mathcal{T}_n}(q) &= \sum_{k=1}^{n} \binom{\frac{q-1}{2}}{k} \times 2^k \times k!S(n,k) + n \times \sum_{k=1}^{n-1} \binom{\frac{q-1}{2}}{k} \times 2^k \times k!S(n-1,k) \\ &= \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{i=1}^{k} (q-(2i-1)). \end{aligned}$$

Since $\chi_{\mathcal{T}_n}$ is a polynomial, we get the required result. □

Remark 6.1. Note that the absolute value of the coefficient of t^j in $(t-1)(t-3)\cdots(t-(2k-1))$ counts the number of signed permutations on $[k]$ with j odd cycles (See A028338 in [16]). Let us denote that number by $a(k,j)$.

Using this we get a compact expression for the coefficient of t^j in $\chi_{\mathcal{T}_n}(t)$ as

$$\sum_{k=j}^{n} (S(n,k) + nS(n-1,k))a(k,j).$$

Corollary 6.2. *The characteristic polynomial of $\mathcal{BT}_n$ has the following form:*

$$\chi_{\mathcal{BT}_n}(t) = \sum_{k=1}^{n} (S(n,k) + nS(n-1,k)) \prod_{i=1}^{k} (t-(2i+1)).$$

Remark 6.2. We can also derive an expression for the exponential generating function for the characteristic polynomial. Using Problem 25(c) of [20, Lecture 5] we get

$$\sum_{n\geq 0} \chi_{\mathcal{BT}_n}(t)\frac{x^n}{n!} = (1+x)(2e^x-1)^{\frac{(t-3)}{2}}.$$

The generating function for the number of regions is

$$\sum_{n\geq 0} r(\mathcal{BT}_n)\frac{x^n}{n!} = \frac{e^{2x}(1-x)}{(2-e^x)^2}.$$

Remark 6.3. Just as in Remark 6.1 we give here a compact expression for the coefficient of t^j in $\chi_{\mathcal{BT}_n}(t)$ as

$$\sum_{k=j}^{n}(S(n,k)+nS(n-1,k))b(k,j)$$

where $b(k,j)$ is the coefficient of t^j in $(t-3)(t-5)\cdots(t-(2k+1))$. It can be shown that $b(k,j) = -\sum_{i=0}^{j} a(k+1,i)$ where $a(k,j)$ is defined in Remark 6.1.

For the sake of completeness we enumerate the coefficients of the characteristic polynomial for smaller values of n (see Table 6.1). Song, in [17], also computed the characteristic polynomial for $n \leq 10$, however there are typos in all the expressions for $n \geq 4$, consequently the region numbers are wrong. We also note here that the sequence of number of regions of $\mathcal{BT}_n$ is not listed in the OEIS [16].

n	$\chi_{\mathcal{BT}_n}(t)$	$r(\mathcal{BT}_n)$
2	t^2-5t+6	12
3	$t^3-9t^2+27t-27$	64
4	$t^4-14t^3+75t^2-181t+165$	436
5	$t^5-20t^4+165t^3-695t^2+1480t-1263$	3624
6	$t^6-27t^5+315t^4-2010t^3+7320t^2-14284t+11559$	35516
7	$t^7-35t^6+546t^5-4865t^4+26460t^3-87010t^2+158753t-122874$	400544
8	$t^8-44t^7+882t^6-10402t^5+78155t^4-379666t^3+1154965t^2-1995487t+1486578$	5106180
9	$t^9-54t^8+1350t^7-20286t^6+200025t^5-1331022t^4+5932143t^3-16952157t^2+27979203t-20158695$	72574936
10	$t^{10}-65t^9+1980t^8-36840t^7+459585t^6-3986031t^5+24172575t^4-100548090t^3+272771475t^2-432836011t+302751327$	1137563980

Table 6.1: Characteristic polynomial and the number of regions of $\mathcal{BT}_n$ for $n \leq 10$.

6.2 The signed ordered partitions

We will now look at a larger arrangement which is obtained by adding the hyperplanes $x_i = -\frac{1}{2}$ and $x_i = \frac{1}{2}$ to the type C arrangement. That is, the boxed version of the type C arrangement (see Proposition 6.1) which we denote by $\mathcal{BC}_n$. Namely, the arrangement with hyperplanes:

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \\ x_i - x_j &= 0 \\ x_i &= -\frac{1}{2} \\ x_i &= \frac{1}{2} \end{aligned}$$

for $1 \leq i < j \leq n$. The regions of this arrangement are given by a valid total order on

$$x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n, \frac{1}{2}, -\frac{1}{2}.$$

We will represent such orders by writing i for x_i and $-i$ for $-x_i$.

Example 6.1. $-x_2 < -\frac{1}{2} < -x_4 < x_3 < -x_1 < x_1 < -x_3 < x_4 < \frac{1}{2} < x_2$ is represented as $-2 \quad -\frac{1}{2} \quad -4 \quad 3 \quad -1 \quad 1 \quad -3 \quad 4 \quad \frac{1}{2} \quad 2$.

It can be checked that such an order is valid if and only if:

1. The order on the numbers $1, \ldots, n, -1 \ldots, -n$ is of the form

   $$i_1 \quad i_2 \quad \ldots \quad i_n \quad -i_n \quad \ldots \quad -i_2 \quad -i_1$$

   where $\{|i_1|, \ldots, |i_n|\} = [n]$.

2. $-\frac{1}{2}$ is before $\frac{1}{2}$ and both appear between i_n and $-i_n$ or $-\frac{1}{2}$ is between i_k and i_{k+1} for some $k \in [n-1]$ and $\frac{1}{2}$ is between the corresponding $-i_{k+1}$ and $-i_k$.

We will call such orders sketches. We also write i as $\overset{+}{i}$ and $-i$ as $\overset{-}{i}$ for all $i \in [n]$. Two numbers that are: both before $-\frac{1}{2}$, both between $-\frac{1}{2}$ and $\frac{1}{2}$ or both after $\frac{1}{2}$, are said to be in the same 'portion' of the sketch. The portion between $-\frac{1}{2}$ and $\frac{1}{2}$ is called the middle portion.

Since $\mathcal{BT}_n$ is a sub-arrangement of $\mathcal{BC}_n$, we can define moves on sketches that correspond to changing some the inequality of some hyperplane in $\mathcal{BC}_n$ that is not in $\mathcal{BT}_n$. The moves are:

1. Swapping i_n and $-i_n$ if $-\frac{1}{2}$ is before i_n.

2. Swapping consecutive i and j (as well as $-i$ and $-j$) for some $i, j \in [n]$ if they are in the same portion.

Example 6.2. An example of a series of the above moves is given below:

$$
\begin{array}{lcccccccccc}
 & \overset{-}{2} & -\frac{1}{2} & \overset{+}{4} & \overset{+}{3} & \overset{-}{1} & \overset{+}{1} & \overset{-}{3} & \overset{-}{4} & \frac{1}{2} & \overset{+}{2} \\
\longrightarrow & \overset{-}{2} & -\frac{1}{2} & \overset{+}{4} & \overset{+}{3} & \overset{+}{1} & \overset{-}{1} & \overset{-}{3} & \overset{-}{4} & \frac{1}{2} & \overset{+}{2} \\
\longrightarrow & \overset{-}{2} & -\frac{1}{2} & \overset{+}{3} & \overset{+}{4} & \overset{+}{1} & \overset{-}{1} & \overset{-}{4} & \overset{-}{3} & \frac{1}{2} & \overset{+}{2}.
\end{array}
$$

We will now count the number of regions of $\mathcal{BT}_n$ by counting the number of equivalence classes of sketches where two sketches are said to be equivalent if one can be obtained from the other by a series of moves of the types mentioned above (which is the same as saying the corresponding $\mathcal{BC}_n$ regions are in the same region of $\mathcal{BT}_n$).

Define a 'block' to be a maximal string of numbers in the same portion of a sketch having the same sign. If there is some number in the middle portion of the sketch, we can always make $-i_n$ and $-i_{n-1}$ have the same sign using a move of the first kind. It can be shown that once this is done, two equivalent sketches have the same elements in each block, the same order of the blocks and same signs for the blocks. Using similar logic as Lemma 5.1, it can be shown that equivalence classes of sketches correspond to orders of the following forms, where $B_1, \ldots, B_k$ is a partition of $[n]$ with a sign assigned to each block:

1. $\frac{1}{2} < B_1 < B_2 < \cdots < B_k$ where B_i and B_{i+1} are of opposite signs for all $i \in [k-1]$.
2. $B_1 < \cdots < B_l < \frac{1}{2} < B_{l+1} < \cdots < B_k$ where the size of B_1 is greater than 1 and B_i and B_{i+1} are of opposite signs for all $i \in [l-1]$ and $i \in [l+1, k-1]$.
3. $B_1 < \frac{1}{2} < B_2 < \cdots < B_k$ where the size of B_1 is 1, B_1 and B_2 are of same sign, and B_i and B_{i+1} are of opposite signs for all $i \in [2, k-1]$.

Proposition 6.3. *The total number of orders of the forms mentioned above is*

$$4a(n) + \sum_{k=1}^{n} 4(k! - (k-1)!)(kS(n,k) - nS(n-1,k-1)).$$

Here $a(n)$ is the n^{th} ordered Bell number and $S(n,k)$ are Stirling numbers of the second kind.

Proof. We will count the number of orders of each of the above forms.

1. In the first case, we just have to define an ordered partition of $[n]$ and assign alternating signs to them. The number of ways this can be done is
$$\sum_{k=1}^{n} \sum_{\substack{(a_1,\ldots,a_k) \\ a_1+\cdots+a_k=n}} 2 \times \frac{n!}{a_1! \ldots a_k!} = 2 \times a(n).$$
2. In the second case, we have to choose the element of $[n]$ in B_1 and then define an ordered partition of the remaining $(n-1)$ elements and assign alternating signs to them. Since we

want B_1 and B_2 to have the same sign, we just need to assign a sign to B_2. So, the number of orders of the second type is:

$$n \times \sum_{k=1}^{n-1} \sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n-1}} 2 \times \frac{(n-1)!}{a_1!\dots a_k!} = n \times 2 \times a(n-1).$$

3. In the third case, we consider two sub-cases:

   (a) There is no block after $\frac{1}{2}$. In this case, we have to define an ordered partition of $[n]$ whose first part has size greater that 1 and assign alternating signs to them. The number of ways this can be done is

$$\sum_{k=1}^{n-1} \sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n,\ a_1\neq 1}} 2 \times \frac{n!}{a_1!\dots a_k!} = 2 \times (a(n) - n \times a(n-1))$$

   where the equality is because the number of ordered partitions of $[n]$ with first block having size 1 is $n \times a(n-1)$.

   (b) There is some block after $\frac{1}{2}$. In this case, we have to again define an ordered partition of $[n]$ whose first part has size greater that 1. But we then have to choose a spot between two blocks to place $\frac{1}{2}$ and then choose a sign for the first block and the block after $\frac{1}{2}$. The number of ways this can be done is

$$\sum_{k=1}^{n-1} \sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n,\ a_1\neq 1}} 4 \times (k-1) \times \frac{n!}{a_1!\dots a_k!}.$$

   Making the following substitution for all $k \in [n-1]$

$$\sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n,\ a_1\neq 1}} \frac{n!}{a_1!\dots a_k!} = \sum_{\substack{(a_1,\dots,a_k) \\ a_1+\cdots+a_k=n}} \frac{n!}{a_1!\dots a_k!} - \sum_{\substack{(1,a_2,\dots,a_k) \\ 1+a_2+\cdots+a_k=n}} \frac{n!}{1!a_2!\dots a_k!}$$
$$= k!S(n,k) - n(k-1)!S(n-1,k-1)$$

   we get that the initial expression is the same as

$$\sum_{k=1}^{n} 4(k! - (k-1)!)(kS(n,k) - nS(n-1,k-1)).$$

Adding up the counts made for each form gives us the required result. □

So, from the observations made above, we have proved the following theorem:

Theorem 6.3. *The number of regions of $\mathcal{BT}_n$ is*

$$4a(n) + \sum_{k=1}^{n} 4(k! - (k-1)!)(kS(n,k) - nS(n-1,k-1)).$$

Remark 6.4. It is also possible to get the block order corresponding to a region of $\mathcal{BT}_n$ directly. This can be done by defining an equivalence on $[-n,n] \setminus \{0\}$ induced by the region of $\mathcal{BT}_n$ and then defining an order on the blocks of this equivalence, $-\frac{1}{2}$ and $\frac{1}{2}$. The details of this method can be found in [6].

6.3 The colored threshold graphs

Before defining the colored threshold graphs that are in bijection with the regions of the boxed threshold arrangement, we recall the bijection between regions of the threshold arrangement and labeled threshold graphs.

Definition 6.1. A threshold graph is defined recursively as follows:

1. The empty graph is a threshold graph.
2. A graph obtained by adding an isolated vertex to a threshold graph is a threshold graph.
3. A graph obtained by adding a vertex adjacent to all vertices of a threshold graph is a threshold graph.

Definition 6.2. A labeled threshold graph is a threshold graph having n vertices with vertices labeled distinctly using $[n]$.

Such labeled threshold graphs can be specified by a signed permutation of $[n]$, that is, a permutation of $[n]$ with a sign associated to each number. The signed permutation $i_1\ i_2\ \ldots\ i_n$ would correspond to the labeled threshold graph obtained by adding vertices labeled $|i_1|, |i_2|, \ldots, |i_n|$ in order where a positive i_k means that $|i_k|$ is added adjacent to all previous vertices and a negative i_k means that it is added isolated to the previous vertices. A maximal string of positive numbers or negative numbers in a signed permutation will be called a block.

Example 6.3. The labeled threshold graph associated to the signed permutation on [5] given by: $\overset{-}{2}\overset{-}{3}\overset{+}{1}\overset{+}{4}\overset{-}{5}$ is shown in Figure 6.1.

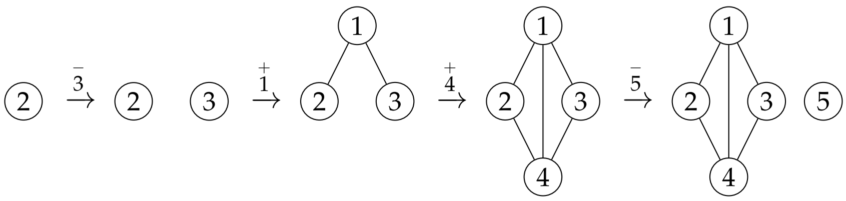


Figure 6.1: Construction of threshold graph corresponding to $\overset{-}{2}\overset{-}{3}\overset{+}{1}\overset{+}{4}\overset{-}{5}$.

The following facts can be verified:

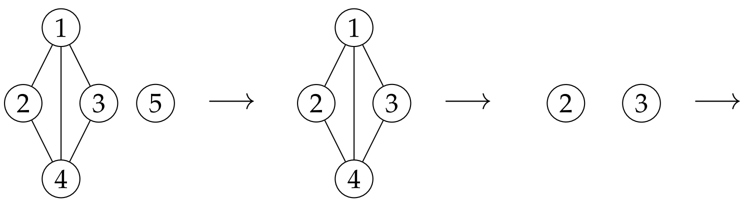
1. The sign of the first number in the permutation does not matter and hence we can make the first block have size greater than 1.
2. Elements in the same block can be reordered.

Hence, labeled threshold graphs can be specified by an ordered partition of $[n]$ with first block size greater than 1 and alternating signs assigned to the blocks. In fact, this association is a bijection.

Example 6.4. The signed permutation $\overset{+}{4}\overset{-}{3}\overset{+}{1}\overset{+}{2}\overset{-}{5}\overset{+}{6}$ would correspond to $\overset{-}{\{3,4\}}\overset{+}{\{1,2\}}\overset{-}{\{5\}}\overset{+}{\{6\}}$.

Given a threshold graph G_1, we can obtain this alternating signed ordered partition of $[n]$ as follows: Since G_1 is a threshold graph, it has at least one isolated vertex or at least one vertex that is adjacent to all other vertices. If it has an isolated vertex, set D_1 to be the set of all isolated vertices, assign it a negative sign and set G_2 to be the graph obtained by deleting all the vertices of D_1 from G_1. If G_1 has at least one vertex adjacent to all other vertices, set D_1 to be the set of all such vertices, assign it a positive sign and set G_2 to be the graph obtained by deleting all the vertices of D_1 from G_1. We repeat this process until we obtain a graph G_k which is complete, in which case we set D_k to be all vertices of G_k and assign it a positive sign, or G_k has no edges, in which case we set D_k to be all vertices of G_k and assign it a negative sign. Then set $B_i = D_{k-i+1}$ and assign it the same sign as D_{k-i+1} for all $i \in [k]$. The signed ordered partition $B_1, \ldots, B_k$ is the one associated to G_1.

Example 6.5. Figure 6.2 shows an example of obtaining the signed blocks from a threshold graph. The corresponding signed ordered partition for this example is: $\overset{-}{\{2,3\}}\overset{+}{\{1,4\}}\overset{-}{\{5\}}$.



$D_1 = \overset{-}{\{5\}}$ $\qquad D_2 = \overset{+}{\{1,4\}}$ $\qquad D_3 = \overset{-}{\{2,3\}}$

Figure 6.2: Obtaining blocks from a threshold graph.

Hence, regions of $\mathcal{T}_n$ and labeled threshold graphs on n vertices are both in bijection with ordered partitions of $[n]$ with first block size greater than 1 and alternating signs assigned to the blocks (see Section 5.1.3). So we obtain a bijection between regions of $\mathcal{T}_n$ and labeled threshold graphs on n vertices. By combining the definitions of the two bijections we see that to a labeled threshold graph on n vertices we assign the region where $x_i + x_j > 0$ if and only if there is an edge between i and j.

This can be proved as follows: If $-B_k < \cdots < -B_1 < B_1 < \cdots < B_k$ is the threshold block order corresponding to some region of $\mathcal{T}_n$, $x_i + x_j > 0$ for some $i \neq j$ in $[n]$ if and only if one of the following hold:

1. $-j$ and i both appear in $B_1, \ldots, B_k$ with $-j$ appearing first.
2. $-j$ appears in $-B_k, \ldots, -B_1$ and i appears in $B_1, \ldots, B_k$.
3. $-j$ and i both appear in $-B_k, \ldots, -B_1$ with $-j$ appearing first.

This is the same as saying: One of the following holds:

1. $-j$ and i both appear in $B_1, \ldots, B_k$ with $-j$ appearing first.
2. i and j both appear in $B_1, \ldots, B_k$.

3. $-i$ and j both appear in $B_1, \ldots, B_k$ with $-i$ appearing first.

But this is precisely the condition for there to be an edge between i and j in the threshold graph corresponding to $B_1 < \cdots < B_k$.

We now move on to the boxed threshold arrangement.

Definition 6.3. A colored threshold graph is defined recursively as follows:

1. The empty graph is a colored threshold graph.
2. A graph obtained by adding an isolated vertex to a colored threshold graph is a colored threshold graph. If there are colored vertices in the initial colored threshold graph, the new vertex should be colored red. If not, the new vertex can be left uncolored or colored red.
3. A graph obtained by adding a vertex adjacent to all vertices of a colored threshold graph is a colored threshold graph. If there are colored vertices in the initial colored threshold graph, the new vertex should be colored blue. If not, the new vertex can be left uncolored or colored blue.

Definition 6.4. A labeled colored threshold graph is a colored threshold graph with n vertices with the vertices labeled distinctly with elements of $[n]$.

Just as for threshold graphs, labeled colored threshold graphs can be represented as a signed permutation. However, we also have to specify if and when the coloring of the vertices starts. This is done by using the symbol $\frac{1}{2}$. Having $\frac{1}{2}$ at the end of the signed permutation means that none of the vertices should be colored.

Example 6.6. The sequence $\overset{+}{2}\frac{1}{2}\overset{+}{1}\overset{+}{3}\overset{-}{4}\overset{-}{5}$ corresponds to the graph shown in Figure 6.3.

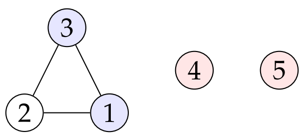


Figure 6.3: Labeled colored threshold graph corresponding to $\overset{+}{2}\frac{1}{2}\overset{+}{1}\overset{+}{3}\overset{-}{4}\overset{-}{5}$.

Using similar observations about these sequences associated to labeled colored threshold graphs as done for labeled threshold graphs, we get that: labeled colored threshold graphs are in bijection with orders of the forms counted in Proposition 6.3. Since these orders also correspond to region of $\mathcal{BT}_n$, we get a bijection between labeled colored threshold graphs with n vertices and regions of $\mathcal{BT}_n$. Just as before, the inequalities describing the region associated to a colored threshold graph are as follows: $x_i + x_j > 0$ if and only if there is an edge between i and j, $-\frac{1}{2} < x_i < \frac{1}{2}$ if i is not colored, $x_i > \frac{1}{2}$ if i is colored blue and $x_i < -\frac{1}{2}$ if i is colored red. Notice that the underlying labeled threshold graph corresponds to the $\mathcal{T}_n$ region that the $\mathcal{BT}_n$ region lies in.

Also, we can see that the bounded regions of $\mathcal{BT}_n$ are in bijection with the regions of $\mathcal{T}_n$. Both are represented by labeled threshold graphs with n vertices. The bounded region of $\mathcal{BT}_n$ corresponding to a region of $\mathcal{T}_n$ is the one satisfying the same inequalities between $x_i + x_j$ and 0 for all $i \neq j$ in $[n]$ and having $-\frac{1}{2} < x_i < \frac{1}{2}$ for all $i \in [n]$.

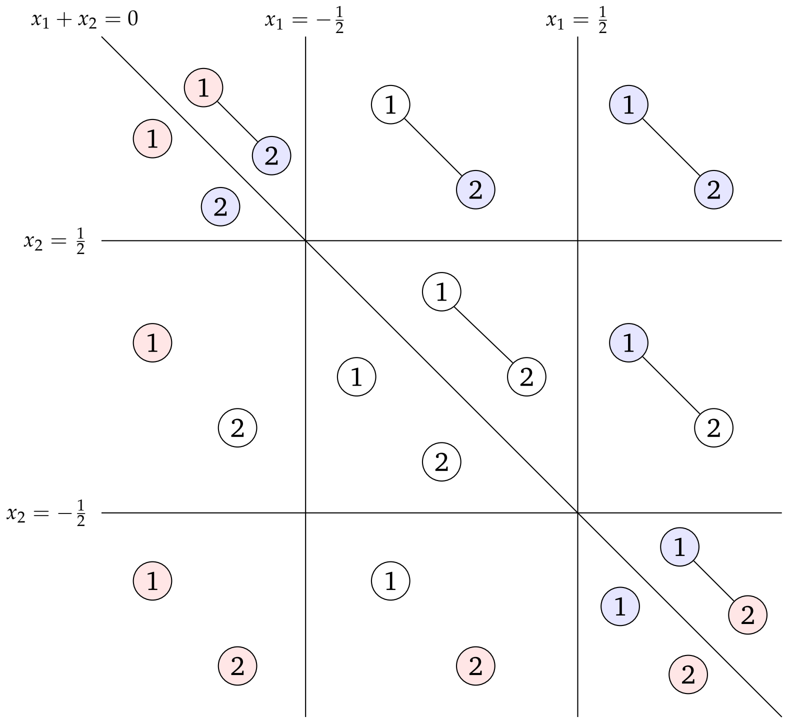


Figure 6.4: Regions of $\mathcal{BT}_2$ represented by labeled colored threshold graphs

Chapter 7

Future Directions

In this chapter, we will mention some possible extensions of the methods used to study hyperplane arrangements mentioned in this thesis.

Extended Catalan. As mentioned in Subsection 5.2.1, it is possible to define sketches that are in bijection with the regions of the extended type C Catalan arrangement. A simpler characterization of these sketches would allow for a broader application of the method of ‘sketches and moves’ to count regions of other arrangements bijectively. It is possible to describe sketches that are in bijection with the extended Catalan arrangements corresponding to the type D, B and BC root systems, using the same method of choosing a canonical sketch from each region. However, counting such sketches would require a better understanding of the symmetric m-sketches.

Other arrangements. The ‘sketches and moves’ method seems to work for other classes of arrangements as well. For example, the proofs in [14] can be viewed as such. Two interesting deformation of the type C Catalan arrangement are the Catalan and Shi threshold arrangement. The characteristic polynomials of these arrangements have been calculated in [15] and [13] using the finite field method. Using moves, it is possible to define sketches (labelled non-nesting partitions) that are in bijection with the regions of the Catalan and Shi threshold arrangement. Similar arguments might be applicable to other arrangements of interest.

Trees. In Chapter 4, we focused on the bijections between regions of deformed braid arrangements, sketches and trees mentioned in [5]. The main result of the paper is one that specifies the number of regions of any deformation of the braid arrangement in terms of *boxed trees*. It would be interesting to see if such a result could be obtained for deformed type C arrangements. Though a bijection between symmetric 1-sketches and certain *symmetric forests* is mentioned in [10], this bijection does not seem to simplify the study of the sketches.

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