Course Project
(Introduction to Reflection Groups)

Three reflection theorem on hyperbolic plane

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Abstract

It is well known that an isometry of an n-dimensional Euclidean space is a product of at most $n+1$ reflections. Aim of my talk is to show that a similar statement is true in the context of hyperbolic plane. I will explain the upper-half plane model and the required concepts from hyperbolic geometry. Finally, conclude that a hyperbolic isometry is a product of at most 3 hyperbolic reflections..

Introduction

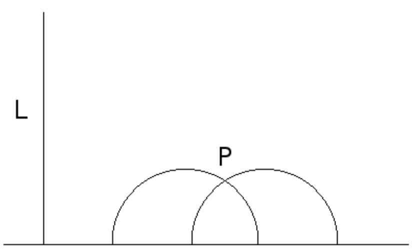
We've proved in the class isometry of S^n and $\mathbb{R}^n$ can be written as at most $n+1$ reflections. In this project we'll prove this theorem for H^n where $n = 2$.

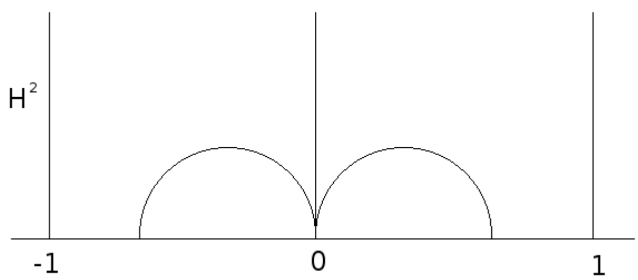
Theorem 1:

Any isometry in H^2 can be written as at most 3 reflections.

Definition of H^2 :

H^2 is the upper half plane. $(y > 0)$





- Here the H^2 lines are circles and semi-circles.
- Two parallel lines meet at ∞.

As in euclidean and spherical geometry, isometry of H^2 are most simply expressed as complex functions. (i.e. functions of $Z = x + iy$)
Here the infinitesimal distance is defined as $dS = |dZ|/ImZ$.
Now we define isometries of H^2 as functions from $H^2 \longmapsto H^2$, listed as below:

(a) $t_\alpha(Z) = \alpha + Z$ for any $\alpha \in \mathbb{R}$.

(b) $d_\rho(Z) = \rho Z$ for any positive $\rho \in \mathbb{R}$.

(c) $\bar{r}_{OY}(Z) = -\bar{Z}$(reflection in the Y-axis).

Here we seem to be lacking rotations. H^2- rotations are best grasped by mapping H^2 onto the open unit disk D^2 , where some rotations materialize as euclidean rotations about the origin.
we can define the H^2 -rotations as $J^{-1}r_\theta J$, where $J\colon H^2 \longmapsto D^2$ can be defined as $J(Z) = \frac{iZ+1}{Z+i}$ and r_θ is a D^2 rotations. All these isometries preserves dS.

Now we are going to discuss an important proposition, which depends on moving an arbitrary H^2- line segment PQ in H^2 to the Y -axis by H^2 -isometries. This can be done by moving P onto the Y -axis by a suitable t_α, then rotating about P, until Q is on the Y-axis. As these H^2 -isometries preserve circles and angles, the H^2-line segment between P and Q must be a part of a circle orthogonal to the real axis and passing through P and Q. This "Circle" is the Y-axis itself.

Proposition :

The H^2- line segment between P and Q is the curve of shortest H^2- length between P and Q.

Proof :

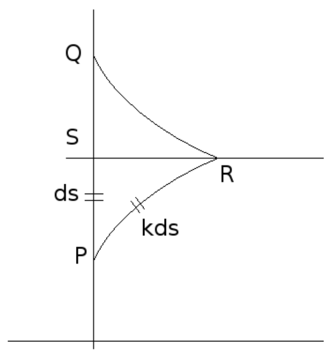
we can assume that the H^2 -line segment PQ is a segment of the Y-axis. Now if C is any other curve from P to Q. H^2-length of $C = \int_C \frac{\sqrt{dx^2+dy^2}}{y} \geqslant \int_P^Q \frac{dy}{y} = H^2$- length of PQ.

Corollary :

If $P, Q, R \in H^2$ (H^2 -length PR) + (H^2 -length RQ) $\geqslant$ (H^2 -length of PQ).

Proof :

We will take $PR \cup RQ$ as the C.
If R is not on the H^2- line through P, Q, which we again take to be on the Y- axis.
Assume the situation in the following picture
Where K $\geqslant sec(RPQ)$.
$\therefore K > 1$.
$\therefore PR > PS$. Similarly $QR > SQ$.
$\therefore PR + QR > PQ$.



Now we want to prove a important lemma. To prove this lemma we want to move arbitrary $P, \acute{P} \in H^2$ to positions which are mirror images of each other in the Y- axis. This can be done by first rotating about P until $\acute{P}$ has the same Y-coordinate as P. Then one applies a suitable t_α to make P and $\acute{P}$ equidistant from the Y -axis.

Lemma :

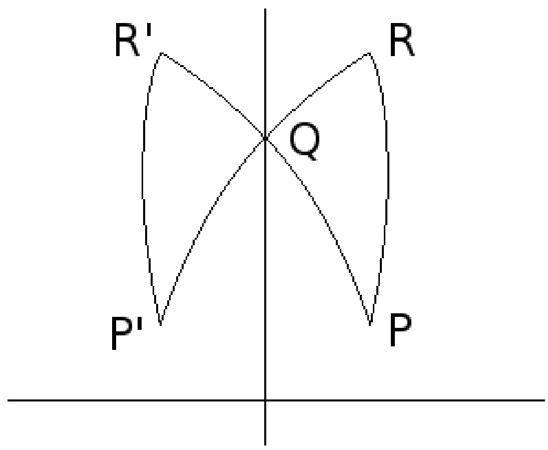
The set of points H^2 -equidistant from two points P, $\acute{P} \in H^2$ is an H^2 - line L, and H^2 - reflection in L exchanges P and $\acute{P}$.

Proof :

By remarks above we can choose P , $\acute{P}$ to be mirror images in the Y -axis , so that reflection $\bar{r}_{OY}$ in the Y-axis exchanges P and $\acute{P}$. Since $\bar{r}_{OY}$ is an H^2 -isometry. Which fixes each point Q on the Y -axis, it follows that any such Q is H^2 - equidistant from P , $\acute{P}$. Thus, the H^2 -equidistant set of P and $\acute{P}$ includes Y-axis which is an H^2-line.

Now suppose that H^2 - equidistant set of P , $\acute{P}$ includes a point R not on the Y -axis. Then the mirror image $\acute{R}$ of R is also H^2 -equidistant from P , $\acute{P}$.

H^2 -length of $\acute{P}$ $\acute{R}$ = H^2 length of PR (by reflection on the Y-axis).
= H^2 length of $\acute{P}R$ (by hypothesis)
= H^2 length of $\acute{P}Q$ + H^2 length of QR
= H^2 length of $\acute{P}Q$ + H^2 length of $Q\acute{R}$
which proves a contradiction due to triangle inequality.



Note :

we can define a line by a set of equidistant points from two given points.

Theorem 2:

Any isometry f of H^2 is the product of one, two or three reflections.

Proof :

Choose three points A, B, C not in a line and consider their f -images $f(A)$,$f(B)$,$f(C)$.

⊙ If two of A, B, C coincide with their f-images, say $A = f(A)$ and $B = f(B)$, then reflection in the line L through A and B must send C to $f(c)$. As C and $f(C)$ are equidistant from $A = f(A)$ and $B = f(B)$. This reflection, therefore sends A, B, C to $f(A)$, $f(B)$, $f(C)$ respectively, and hence coincides with f.

⊙ If one of A, B, C coincide with its f-image, say $A = f(A)$, first perform the reflection $\bar{g}$ in the line M of points equidistant from B and $f(B)$, hence on M, it is fixed by $\bar{g}$, so $\bar{g}$ sends A, B to $f(A)$, $f(B)$. If $\bar{g}$ also sends C to $f(C)$, we are finished. If not, perform another reflection $\bar{h}$ in the line through $f(A)$, $f(B)$ and conclude as above $\bar{h}\ \bar{g}\ (c) = f(C)$. Then $\bar{h}\ \bar{g}$ sends A, B, C to $f(A)$, $f(B)$, $f(C)$.

⊙ Finally if none of A, B, C coincides with their f-images then we perform up to 3 reflections, in the lines L, M, N equidistant from $A = f(A)$, $B = f(B)$, $C = f(C)$ respectively. Then one can easily see similarly the product of these reflections sends A to $f(A)$, B to $f(B)$ and C to $f(C)$.

References

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[3] tur-www1.massey.ac.nz/ ctuffley/slides/threereflections.pdf .