Course Project
(Introduction to Reflection Groups)

Braid Groups and Configuration Spaces

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Abstract

In this article we introduce the notions of Braids group and Configuration space and generalize for an arbitrary Coxeter group.

1 Introduction

To any Coxeter group is associated a topological space called the Configuration space. Trying to find the fundamental group of this leads naturally to the notion of Artin group. The special case of this is the Braid group. This special case can be analyzed using elementary methods in algebraic topology. This report merely states how the connection between Artin groups and Configuration spaces is established. The proofs can be found in the references.

2 Braid Groups

Here X will denote a connected C^∞ manifold, Σ_k will denote the symmetric group of order k and all maps are assumed to be continuous. There would not be any loss of understanding if you assume $X = \mathbb{R}^2$ for this section.

2.1 Topology

Definition 2.1. *For an integer k let*

$$F(X,k) = \{(x_1, \cdots, x_k) \in X^k | i \neq j \implies x_i \neq x_j\}$$

This is called the $k-configuration$ space of X and is given the subspace topology.

There is a natural Σ_k action on $F(X,k)$. Denote the quotient by,

$$SF(X,k) = F(X,k)/\Sigma_k$$

Definition 2.2 (k-Braid groups)**.** *A $k-$ Braid group in X based at $e = (e_1, e_2, \cdots, e_n) \in F(X,k)$ is defined to be equal to $\pi_1(SF(X,k), e)$. The pure braid group is the fundamental group $\pi_1(F(X,k), e)$.*

Because we have assumed that X is connected, one can show that so is $F(X,k)$ and hence $SF(X,k)$, and so the choice of basepoint does not matter. As such we will be lazy and neglect the basepoint.

Note that because $SF(X,k)$ is a Galois cover of $F(X,k)$ with the Galois group being Σ_k we have the short exact sequence

$$1 \to \pi_1(F(X,k) \to \pi_1(SF(X,k) \to \Sigma_k \to 1$$

The elements of $\pi_1(SF(X,k)$ can also be thought of as being represented by k non-intersecting paths *which are always moving forward* in $X \times [0,1]$ which begin and end at the points of e. The pure braid group consists of those paths which start and end at the same point of X. In this interpretation the group multiplication just becomes concatenation. This interpretation is more intuitive though is not so easy to work with.

The special case $X = \mathbb{R}^2$ is the **Classical Braid group**.

2.2 Algebra

As it turns out it is possible to explicitly write down the generators of $\pi_1(F(X,k))$ and $\pi_1(SF(X,k))$. We will describe the less complicated ones, that of the impure Braid group $\pi_1(SF(X,k))$.

We will use the second interpretation of the Braid group. Denote by σ_i the element of $\pi_1(SF(X,k))$ which permutes the i^{th} and the $i+1^{th}$ strand. Then it is easy to see that these operations generate $\pi_1(SF(X,k))$. So the real question becomes what are the relations between these. With a little imagination one can see that the following relations hold:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$
2. $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$

Definition 2.3. *Define the group B_n having the following presentation,*

$$< \sigma_1, \cdots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \textit{ for } |i-j| > 1, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \textit{ for } |i-j| = 1 >$$

Then by the above observation we have a natural surjection $\psi : B_n \to \pi_1(SF(X,k))$.

Theorem 2.4. *ψ is an isomorphism.*

The proof of this is elementary but tedious. The idea is to find a relationship between $SF(X,k)$ and $SF(X,k-1)$. Unfortunately such a relation is not easy to find. On the other hand we have a canonical projection map $F(X,k) \to F(X,k-1)$. It is easy to see that this map is a fiber bundle with the fiber being homotopy equivalent to wedge of k circles. The fibration long exact sequence applied to this fiber bundle then gives the answer. For details see, [2, 4].

3 Artin Groups and Configuration Spaces

What we did above can be done in a vast generality for arbitrary Coxeter groups.

Let W denote a finitely generated Coxeter group with generators $\sigma_1, \cdots, \sigma_n$ and Coxeter matrix $m_{i,j}$.

Definition 3.1 (Artin group)**.** *The Artin group $A(W)$ associated to the Coxeter group W is the group having the following presentation,*

$$< \sigma_1, \cdots, \sigma_n | (\sigma_i \sigma_j)^{m_{i,j}/2} = (\sigma_j \sigma_i)^{m_{i,j}/2} \text{ if } m_{i,j} \text{ even },$$
$$(\sigma_i \sigma_j)^{m_{i,j}-1/2} \sigma_i = (\sigma_j \sigma_i)^{m_{i,j}-1/2} \sigma_j \text{ if } m_{i,j} \text{ even } >$$

Note that this notation does not force the generators to be reflections.

The braid group B_n is precisely the Artin group associated to the symmetric group Σ_n.

We need to reinterpret the configuration space $F(\mathbb{R}^2, k)$ to be able to generalize it. $FS(\mathbb{R}^2, k) \subset \mathbb{R}^{2k}$ which can be thought as the subset of $\mathbb{R}^k \times \mathbb{R}^k$ with the space $D \times D$ removed, where the space $D \subset \mathbb{R}^k$ contains points with not all coordinates distinct.

It is known that W can be thought of as a reflection group acting essentially (i.e. there is no invariant subspace) on $\mathbb{R}^k$ for some suitable k (see [5]). Let the H be the union of all the hyperplanes corresponding to the reflections in W, this is the D in the case of Σ_n. Then define,

Definition 3.2 (Configuration space)**.** *The configuration space $N(W)$ is defined to be the space $(\mathbb{R}^k \times \mathbb{R}^k) \backslash (H \times H)$. There is a natural Galois action of W on $N(W)$, the quotient space $N(W)/H$ is denoted $M(W)$.*

The corresponding braid groups will then be $\pi_1(N(W))$ and $\pi_1(M(W))$.

So what are the corresponding generators and relations for these?

Theorem 3.3.

$$\pi_1(M(W)) = A(W)$$

Whether the higher homotopy groups are trivial or not is still an open problem. For details see, [1].

References

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[4] Hatcher, A., Algebraic topology, Cambridge University Press ISBN: 0521541867

[5] Borovik, A., Mirrors and reflections the geometry of finite reflection groups, Springer ISBN: 0387790659