Introduction to Manifolds

Assignment 1
Due Date: 16/08/2018

Problem 1: In each of the following a function f is given, find a generic expression for its derivative Df, then determine when the derivative is non-singular. Finally, for the given subset S of the domain sketch its image $f(S)$.

1. $f(r, \theta) = (r\cos\theta, r\sin\theta)$ and $S = [1, 2] \times [0, 2\pi]$.
2. $f(x, y) = (x^2 - y^2, 2xy)$ and $S = \{(x, y) \mid x^2 + y^2 \leq a^2, x \geq 0, y \geq 0, a \geq 0\}$.
3. $f(x, y) = (e^x \cos y, e^x \sin y)$ and $S = [0, 1] \times [0, 2\pi]$.
4. $f(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$ and $S = [1, 2) \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Problem 2: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear function.

1. Prove that
$$\lim_{(h,k)\to 0} \frac{\|f(h, k)\|}{\|(h, k)\|} = 0,$$
and that f is differentiable everywhere in the domain.
2. Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.
3. Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be two differentiable maps and $h : \mathbb{R} \to \mathbb{R}$ is defined by setting $h(t) = \langle f(t), g(t)\rangle$ (the standard inner product on $\mathbb{R}^n$). Prove that h is differentiable everywhere and find $h'(a)$.

Problem 3: Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be nonempty, diffeomorphic subsets. Then prove the following.

1. If U and V both are open then $m = n$.
2. If $m = n$ and U is open then V is also open.

Problem 4: Let $X = [-1, 1]$ and $Y = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{2}\} \cup \{(0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{1}{2}\}$. Are X and Y diffeomorphic? Justify.

Problem 5: Show that the unit sphere S^n in $\mathbb{R}^{n+1}$ is a smooth n-manifold using stereographic projection as a coordinate chart.