Introduction to Manifolds

Mid Semester Exam
Solutions

Problem 1: Denote by $G(2, \mathbb{R}^3)$ the set of all planes (passing through the origin) in $\mathbb{R}^3$. Show that $G(2, \mathbb{R}^3)$ is a 2-manifold and determine its homeomorphism type.

Solution: For every plane passing through the origin there is a unique line orthogonal to it. This correspondence gives a clear bijection between lines and planes in $\mathbb{R}^3$. In other words, $G(2, \mathbb{R}^3)$ is in bijection with $\mathbb{RP}^2$. Hence, we can put the same smooth structure as $\mathbb{RP}^2$ on $G(2, \mathbb{R}^3)$.

Problem 2: Consider the map $F : \mathbb{RP}^1 \to \mathbb{R}^2$ defined by

$$x = \frac{2u_1u_2}{u_1^2 + u_2^2}, \quad y = \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2}$$

where $[u_1 : u_2] \in \mathbb{RP}^1$ and $(x, y) \in \mathbb{R}^2$. Show that F is well defined, smooth and a diffeomorphism onto S^1.

Solution: It is fairly straightforward to show that the given map is well defined and that its image is contained in S^1. Now for smoothness, consider the chart on $\mathbb{RP}^1$ on which $u_1 \neq 0$. Then we get the composition

$$s \mapsto [1 : s] \mapsto \left(\frac{2s}{1 + s^2}, \frac{1 - s^2}{1 + s^2} \right)$$

which is clearly smooth; moreover, a similar argument works for the other chart.
Now in order to show that it is a bijection, a straightforward manipulation shows it is one-one. Observe that for $(\cos\theta, \sin\theta) \in S^1$ the point $[\sin\theta/2 : \cos\theta/2] \in \mathbb{RP}^1$ is its inverse image; implying F is onto.
The last step is to show that F is an immersion. Now in the first chart the differential of F takes the following form

$$\begin{bmatrix} \frac{2(1 - s^2)}{(1 + s^2)^2} & \frac{-4s}{(1 + s^2)^2} \end{bmatrix}.$$

The above matrix is zero if and only if $s = 0$ and $1 - s^2 = 0$ which is a absurd. A similar argument works for the second chart.

Problem 3: Show that the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is diffeomorphic to the infinite cylinder $\mathbb{R} \times S^1$.

Solution: It is convenient to consider the polar coordinates on $\mathbb{R}^2$ and also to consider $\mathbb{R} \times S^1$ as a subset of $\mathbb{R}^3$. Consider the map $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3$ given by

$$(r, \theta) \mapsto (\ln r, \cos\theta, \sin\theta).$$

Its image is the infinite cylinder and the map is a bijection onto the image. A simple calculation shows that the derivative matrix is always rank 2 hence f is also an immersion.

Problem 4: For $a \in \mathbb{R}$ consider the set

$$M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = a\}.$$

For what values of a the set M_a is a manifold? What is its dimension? Explain.

Solution: Use the regular value theorem to conclude that for $a \neq 0$ the hyperboloid (of one sheet if $a > 0$ and of two sheets if $a < 0$) is a regular codimension-1 submanifold of $\mathbb{R}^3$. For $a = 0$, we get the cone which fails to be locally Euclidean at the origin.

Problem 5: Explicitly determine $T_p(S^1)$ at an arbitrary point $p = (a, b) \in S^1$.

Solution: We know that $S^1 \subset \mathbb{R}^2$ so for any $p \in S^1$ the tangent space $T_p(S^1)$ is a linear subspace of $T_p(\mathbb{R}^2) = \mathbb{R}^2$. Let $f(x, y) = x^2+y^2-1$. Then S^1 is the regular zero set $f^{-1}(0)$. Let c be a curve in S^1 starting at $p = (a, b)$ with $c'(0)$ as a tangent vector in $T_p(S^1)$. We know that $f(c(t)) = 0$ for all t's. Taking the derivative (invoking the chain rule) followed by a simple calculation shows that the equation of $T_p(S^1)$ in $\mathbb{R}^2$ is $ax + by = 0$.

Problem 6: Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the inversion map, $f(A) = A^{-1}$. Show that

$$f_{*,I} : \mathrm{M}(n, \mathbb{R}) \to \mathrm{M}(n, \mathbb{R}) \text{ is given by } X \mapsto -X.$$

Solution: Let g be the constant map $A \mapsto AA^{-1} = I$ and let c be a curve starting at I and $c'(0) = X$. Since $g(c(t)) = I$ for all t we get

$$0 = \frac{d}{dt} g(c(t))|_{t=0} = c'(0)I + c(0)f_{*,I}(c'(0)).$$

Bonus problem: For an arbitrary $C \in \mathrm{GL}(n, \mathbb{R})$ determine the linear map $f_{*,C} : T_C\mathrm{GL} \to T_{C^{-1}}\mathrm{GL}$. (Hint: Use $f \circ l_C = r_{C^{-1}} \circ f$, where l is the left multiplication and r is the right multiplication.)

Solution: Observe that the hint basically says that $(CA)^{-1} = A^{-1}C^{-1}$. Now applying the chain rule to the differentials of the above composition we get

$$f_{*,C} \circ (l_C)_{*,I} = (r_{C^{-1}})_{*,I} \circ f_{*,I}.$$

Since both the right and left multiplications are diffeomorphisms the induced derivatives are linear isomorphisms. Hence we have

$$f_{*,C}(X) = (r_{C^{-1}})_{*,I} \circ f_{*,I} \circ (l_C)^{-1}_{*,I}(X) = -C^{-1}XC^{-1}.$$