Introduction to Manifolds

Mid Semester Exam
Date: 28/09/2017

1. Time allowed: 3 hours.
2. Each problem is worth 10 points.
3. The bonus problem is worth 15 points.

Problem 1: Denote by $G(2, \mathbb{R}^3)$ the set of all planes (passing through the origin) in $\mathbb{R}^3$. Show that $G(2, \mathbb{R}^3)$ is a 2-manifold and determine its homeomorphism type.

Problem 2: Consider the map $F : \mathbb{RP}^1 \to \mathbb{R}^2$ defined by

$$x = \frac{2u_1u_2}{u_1^2 + u_2^2}, \quad y = \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2}$$

where $[u_1 : u_2] \in \mathbb{RP}^1$ and $(x, y) \in \mathbb{R}^2$. Show that F is well defined, smooth and a diffeomorphism onto S^1.

Problem 3: Show that the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is diffeomorphic to the infinite cylinder $\mathbb{R} \times S^1$.

Problem 4: For $a \in \mathbb{R}$ consider the set

$$M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = a\}.$$

For what values of a the set M_a is a manifold? What is its dimension? Explain.

Problem 5: Explicitly determine $T_p(S^1)$ at an arbitrary point $p = (a, b) \in S^1$.

Problem 6: Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the inversion map, $f(A) = A^{-1}$. Show that

$$f_{*,I} : \mathrm{M}(n, \mathbb{R}) \to \mathrm{M}(n, \mathbb{R}) \text{ is given by } X \mapsto -X.$$

Bonus problem: For an arbitrary $C \in \mathrm{GL}(n, \mathbb{R})$ determine the linear map $f_{*,C} : T_C\mathrm{GL} \to T_{C^{-1}}\mathrm{GL}$. (Hint: Use $f \circ l_C = r_{C^{-1}} \circ f$, where l is the left multiplication and r is the right multiplication.)