Introduction to Manifolds

Assignment 1
Due Date: 24/08/2017

Problem 1: In each of the following a function f is given, find a generic expression for its derivative Df, then determine when the derivative is non-singular. Finally, for the given subset S of the domain sketch its image $f(S)$.

1. $f(r, \theta) = (r \cos \theta, r \sin \theta)$ and $S = [1, 2] \times [0, 2\pi]$.
2. $f(x, y) = (x^2 - y^2, 2xy)$ and $S = \{(x, y) \mid x^2 + y^2 \leq a^2, x \geq 0, y \geq 0, a \geq 0\}$.
3. $f(x, y) = (e^x \cos y, e^x \sin y)$ and $S = [0, 1] \times [0, 2\pi]$.
4. $f(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$ and $S = [1, 2) \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Problem 2: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear function.

1. Prove that if f is bilinear, then

$$\lim_{(h,k)\to 0} \frac{\|f(h, k)\|}{\|(h, k)\|} = 0,$$

   and that f is differentiable everywhere in the domain.

2. Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.
3. Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be two differentiable maps and $h : \mathbb{R} \to \mathbb{R}$ is defined by setting $h(t) = \langle f(t), g(t) \rangle$ (the standard inner product on $\mathbb{R}^n$). Prove that h is differentiable everywhere and find $h'(a)$.

Problem 3: Let $\{V_1, \dots, V_k\}$ be a collection of finite-dimensional vector spaces such that $\dim V_i = n_i$ for every i. Let $f : V_1 \times \cdots \times V_k \to \mathbb{R}^p$ be a multilinear map (i.e., linear in each variable).

1. Show that for $a := (a_1, \dots, a_k)$ and $h := (h_1, \dots, h_k)$ we have

$$\lim_{h\to 0} \frac{\|f(a_1, \dots, h_i, \dots, h_j, \dots, a_k)\|}{\|h\|} = 0.$$

2. Prove that f is differentiable everywhere.

3. Find the matrix $Df(a)$ and also $Df(a)(x)$.

Problem 4: Let $M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries.

1. Prove that the function $\det : M(n, \mathbb{R}) \to \mathbb{R}$ which sends a matrix to its determinant is differentiable and find $D\det(A)(X)$.

2. Find $D\det(I_n)(X)$ and use it to show that the trace map is differentiable.

Problem 5: Compute the derivative of the following functions.

1. $f : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $f(A, B) = A + B$.

2. $g : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $g(A, B) = AB$.

3. $h : M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $h(A) = A^2$.

4. $j : M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $j(A) = A^t$.