Introduction to Manifolds

Assignment 2
Due Date: 07/09/2017

Problem 1: Let X be a finite-dimensional (abstract) smooth manifold. Show that with respect to the manifold topology on X the coordinate maps $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ are homeomorphisms.

Problem 2: Let $U_N \subset S^2$ be the complement of $(0,0,1)$. Prove that U_N is diffeomorphic to $\mathbb{R}^2$ by exhibiting an explicit map and its inverse. Repeat this for the complement of $(0,0,-1)$, i.e., show that it is a coordinate neighborhood. Conclude that these charts give S^2 a structure of smooth 2-manifold.

Problem 3: Show that it is not possible to cover the unit d-sphere in $\mathbb{R}^{d+1}$ by a single coordinate chart.

Problem 4: Let $B_a = \{x \in \mathbb{R}^n \mid \|x\|^2 < a^2\}$ be an open ball of radius a in $\mathbb{R}^n$. Show that the map

$$x \mapsto \frac{ax}{\sqrt{a^2 - \|x\|^2}}$$

is a diffeomorphism of B_a onto $\mathbb{R}^n$.

Problem 5: Prove that the union of the two coordinate axes in the Euclidean plane is not a manifold.

Problem 6: Let M and N be two smooth manifolds of dimension m and n respectively. Prove that the cartesian product $M \times N$ is a smooth manifold of dimension $m + n$.