Reflection Groups

Homework 2

Due date: 01/02/2013

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1 Arrangements of lines

Definition 1.1. An arrangement of lines $\mathcal{L} = \{l_1, \ldots, l_n\}$ is a finite collection of straight lines in $\mathbb{A}^2$ satisfying following conditions:

1. no two lines are parallel;
2. not all lines pass through a single point (i.e., $\bigcap_{i=1}^n l_i = \emptyset$).

An arrangement $\mathcal{L}$ decomposes $\mathbb{A}^2$ into a finite number of vertices, edges and cells whose numbers are denoted by $f_0(\mathcal{L}), f_1(\mathcal{L})$ and $f_2(\mathcal{L})$ respectively.

Definition 1.2. Two line arrangements are isomorphic if and only if there is an incidence preserving one-to-one correspondence between vertices, edges and cells of one arrangement and those of the other arrangement. The number of isomorphism types of arrangements of n lines is denoted by $c(n)$.

A *simple arrangement* is the one in which every vertex is an intersection of exactly 2 lines. The number of isomorphism types of simple arrangements is denoted by $c^s(n)$. A *simplicial arrangement* is the one in which all the cells are triangles (bounded or otherwise). The number of their isomorphism types is denoted by $c^\Delta(n)$.

For example, an arrangement of 3 lines is both simple and simplicial. In fact, $c(3) = c^s(3) = c^\Delta(3) = 1$.

1. For a simple arrangement of n lines find a formula, in terms of n, for f_i's.
2. For $4 \leq n \leq 6$ determine $c(n)$. Also draw the picture of a representative of each isomorphism type.
3. For $4 \leq n \leq 7$ determine $c^s(n)$. Also draw the picture of a representative of each isomorphism type.
4. For $4 \leq n \leq 10$ determine $c^\Delta(n)$. Also draw the picture of a representative of each isomorphism type.
5. **(Bonus)**. Is it possible to classify simplicial arrangements ? If yes then state your classification. (i.e., try to find patterns in simplicial arrangements). Prepare a table which illustrates, for each isomorphism type of simplicial arrangement you have obtained, the numbers n, f_0, f_1, f_2 and the number of bounded cells.
6. **(Bonus)**. Formulate a reasonable conjecture regarding $c^\Delta(n)$ for all n.

2 Groups of symmetries

Definition 2.1. For a polyhedron $\Delta \subset \mathbb{A}^n$, the *group of symmetries* $\mathrm{Sym}\Delta$ consists of isometries of $\mathbb{A}^n$ that map Δ onto Δ.

1. Give examples of polytopes in $\mathbb{A}^3$ such that
   (a) $\mathrm{Sym}\Delta$ acts transitively on the set of vertices but is intransitive on the set of faces.
   (b) $\mathrm{Sym}\Delta$ acts transitively on the set of faces but is intransitive on the set of vertices.
   (c) $\mathrm{Sym}\Delta$ is transitive on the set of edges but is intransitive on the set of faces.
2. Prove that symmetry group of a polytope is finite.
3. Construct a polyhedron with an infinite symmetry group.