Topics in Combinatorics

Assignment 4
Due Date: 19/03/2018

Recall the Catalan arrangement in $\mathbb{R}^n$:

$$\text{cat}_n := \{x_i - x_j = -1, 0, 1 \mid 1 \le i < j \le n\}.$$

Denote by $R_0(\text{cat}_n)$ the set of all regions that satisfy $x_1 > x_2 > \cdots > x_n$. We have seen in class that this set is in bijection with the lattice (Dyck) paths from $(0,0)$ to (n,n) and hence its cardinality is C_n; the n-th Catalan number. Given below is a list of various combinatorially defined sets; for each such set describe an explicit bijection between this set and $R_0(\text{cat}_n)$.

1. An n-mountain range is a piecewise linear path in $\mathbb{R}^2$ from the origin to $(2n, 0)$ subject to the following two conditions -
   - the path can touch the x-axis but can't cross it and,
   - from the point (x, y) one can either climb up to the point $(x+1, y+1)$ or climb down to the point $(x+1, y-1)$.

   Show that the set of all n-mountain ranges is in bijection with $R_0(\text{cat}_n)$.

2. A sequence consisting of some open parentheses and some closed parentheses is said to be balanced if each opening symbol has a corresponding closing symbol and the pairs of parentheses are properly nested. Now consider the set of all balanced parentheses sequences of length $2n$.

3. Draw n points on a line (equally spaced), label them $1, \ldots, n$ and call them vertices. Connect two vertices labeled i, j by an edge that does not drop below the line such that no two edges intersect except possibly at vertices and at each vertex, the edges all exit in one direction (all exit to the left or all exit to the right). Call such a picture (graph, to be precise) a noncrossing, alternating tree on n vertices ($\mathcal{NA} - trees$). Show that the set of all $\mathcal{NA}$-trees on $n+1$ vertices are in bijection with $R(\text{cat}_n)$

4. For a simple convex n-gon by a chord we mean either an internal diagonal joining non-adjacent vertices or a side joining two adjacent vertices. A polygon with a set distinguished chords is called chorded. Now consider the set of chorded $(n+1)$-gons with n-chords such that the chords do not intersect except perhaps at the vertices and no vertex in $[n+1]$ is adjacent to both a smaller and a larger vertex. (The vertices of the polygon are labeled $1, \dots, n+1$.)

5. The set of all bracketings of a string of $n+1$ factors $x_0, \dots, x_n$ subject to a nonassociative binary operations. For example,

$$((x_0x_1)x_2)x_3, \quad (x_0x_1)(x_2x_3), \quad (x_0(x_1x_2))x_3, \quad x_0((x_1x_2)x_3), \quad x_0(x_1(x_2x_3)).$$