Topics in Combinatorics

Assignment 1
Due Date: 12/01/2018

Problem 1: Let $\mathscr{A}$ be an arrangement of n ($n \geq 1$) lines and f_2 be the number of regions/ chambers of $\mathscr{A}$. Denote by p the maximal number of parallel lines in $\mathscr{A}$, by q the maximal number of concurrent lines in $\mathscr{A}$. Finally, for $i \geq 2$, let t_i be the number of vertices incident with i lines. Now prove the following.

1. $f_2 \geq (p+1)(n-p+1)$.
2. $f_2 \geq q(n-q+2)$.
3. Construct two arrangements $\mathscr{A}$ and $\mathscr{B}$ such that $f_2(\mathscr{A}) = (p+1)(n-p+1)$ and $f_2(\mathscr{B}) = q(n-q+2)$.
4. $f_2 = n+1+\sum_{i=2}^{q}(i-1)t_i$

Problem 2: Prove that the number f_2 can not belong to the following intervals:

1. $(n+1, 2n)$ for $n \geq 3$,
2. $(2n, 3n-3)$ for $n \geq 5$.

Problem 3: Find the maximum possible value of f_2 when, n, p are fixed and when n, q are fixed.

Problem 4: Given n, p, where $1 \leq p \leq n$, define the following numbers

$$\beta(n,p) := (p+1)(n-p+1) + \binom{n-p}{2} \qquad \alpha(n,p) := \beta(n,p) - \min\{p, \binom{n-p}{2}\}.$$

For any integer f, where $\alpha(n,p) \leq f \leq \beta(n,p)$, describe a construction of an arrangement $\mathscr{A}$ of n lines such that $f_2(\mathscr{A}) = f$.