Dishant Pancholi

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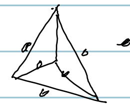
Review :

We showed that any orientable 3-mfld M^3

$$M^3 = H_1 \cup_f H_2$$

where H_i is a handle body.

Proof

Take a triangulation → 

Take 1- skeleton

K_1 - . .

~~Take K_2~~ Take – 1st barycentric subdivision

K_2 = all 1-dim'l simplices not meeting K_1.

Take thickened the 

Take 2nd barycentric subdivision & look at $N(K_1)$ and $N(K_2)$ which are simplicial nbhd of K_1.

Equivalently $K_2 \sim$ all 1-simplices satisfying the following property: If σ is a simplex in $\tilde{\tau}$ – 2nd barycent subdivn ~~then~~

* such that K_2 is a face of σ then $\sigma \cap K_1 = \phi$.

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Koszul complexes
((Aprodu – Nagel) Serre local alg))

Let k be a field, E f.d.v.s $/k$ $\dim_k E = r$.

Let $s \in$ ~~$V^* = \mathrm{Hom}_k(V, k)$~~ $E^* = \mathrm{Hom}_k(E, k)$
Using this can define a contraction map

$$i_s : \Lambda^* E \longrightarrow \Lambda^* E, \quad (\text{of degree } -1)$$

defined inductively

$$i_s : \Lambda^k E \longrightarrow \Lambda^{k-1} E$$

$$e_0 \wedge e_1 \wedge \cdots \wedge e_{k-1} \longmapsto s(e_0)(e_1 \wedge \cdots \wedge e_{k-1}) - e_0 \wedge i_s(e_1 \wedge \cdots \wedge e_{k-1})$$

Fact : $i_s \circ i_s = 0$.

Hence get complex

$$K_\bullet(s) \quad 0 \to \Lambda^r E \to \Lambda^{r-1} E \to \cdots \to \Lambda^0 E \to E \otimes k \to 0$$

Fact : $K_\bullet(s)$ is exact.

Except for the statement about exactness, everything else works even if we had taken k to be Noeth. local ring, and E to be a free k-module of finite rank r. Let $e_1, \ldots, e_r$

be a basis of E and write $s = \sum_i f_i e_i^*$, where $f_i \in k$.

<u>Fact</u>: If f_1 is a NZD on k and f_i is a NZD on $k/f_1,\dots,f_{i-1}$ $\forall\ 2 \le i \le r$, then $K_\bullet(s)$ is a-cyclic. Conversely if $\langle f_1,\dots,f_r\rangle \ne k$ (k local) then the converse holds, i.e. $K_\bullet(s)$ is acyclic.

[<u>Aside</u>: A-complex $C_\bullet$ is acyclic if $C_i = 0$ $i \ge 1$, and $H_i(C_\bullet) = 0$ for $i \ne 0$].

<u>Rmk</u>: Say that $(f_1,\dots,f_r)$ has <u>depth</u> (sometimes <u>grade</u>) r in k if f_1 is a NZD on k and f_i is a NZD on $k/(f_1,\dots,f_{i-1})$ $i = 2,\dots,r$.

Since k is Noeth. local, $(f_1,\dots,f_r)$ has depth r implies $\operatorname{Codim}(f_1,\dots,f_r) = r$.
If k is a regular local ring (or even Cohen-Macaulay) then every k ideal I will satisfy $\operatorname{Codim} I = \operatorname{depth} I$.

Now let X be a smooth variety, and E a rank r vector bundle on X. Suppose there is a non-zero section s of E^*. We can repeat the construction from above to get a complex (again called the Koszul complex)

$$K_\bullet(s).\quad 0 \to \Lambda^r E \to \Lambda^{r-1} E \to \dots \to \Lambda^2 E \to E \to \mathcal{O}_X \to 0.$$

Denote the zero locus of s by $Z(s)$.
In physical (set theoretic) terms:

$$Z(s) = \{x \in X \mid s(x) = 0 \text{ in the fibre } (E^*)_x\}.$$

$K_\circ(s)$ is acyclic if and only if for every $x \in Z(s)$, $Z(s)$ is defined (locally) in $\mathcal{O}_{X,x}$ ~~form~~ by a regular sequence, i.e. $Z(s)$ ~~is~~ has codim r in $\mathcal{O}_{X,x}$ (it is already defined by ~~at~~ r equns).

<u>Back to Grassmannians</u>:

Recall the tautological sequence

$$0 \to R \to E \times \mathrm{Grass}(r, E) \to Q \to 0 \quad \text{tautological sequence}$$

Here $\mathrm{rk}\, E = n$, and $\mathrm{rk}\, R = r$.

<u>Proposition</u>: $\mathcal{O}_R$ is resolved as an $\mathcal{O}_{E \times \mathrm{Grass}(r,E)} = \mathcal{O}_{E\times G}$

[where (for today) $G = \mathrm{Grass}(r, E)$]

by the following Koszul complex

$$K_\circ(Q^*) : \Lambda^{n-r}(p^* Q^*) \to \cdots \to \Lambda^2(p^* Q^*) \to p^* Q^* \to \mathcal{O}_{E \times G} \to 0$$

$$p : E \times G \to G.$$

Proof: ~~Let~~ First here is Balaji's comment:

Let have

$$\begin{tikzcd} p^*Q \arrow[r] \arrow[d] & Q \arrow[d] \\ E\times G \arrow[r, "p"'] \arrow[ur] \arrow[u, bend left] & G \end{tikzcd}$$

More precisely

$$\begin{tikzcd} E\times G \arrow[r] \arrow[dr] & Q \arrow[d] \\ & G \end{tikzcd}$$

gives ~~#~~ rise to a section $E\times G \to p^*Q$ of $p^*Q \to E\times G$.

~~Here is the more local proof proo~~

This gives us the section of p^*Q ~~on~~ from which the Koszul $K_\bullet(Q^*)$ is built.

Here is the way to get it via equations etc (a la Weyman).

Note ~~Let~~ $\{U_I : |I| = r \quad I \subseteq \{1,\dots,n\}\}$ form an affine open cover of G.

The eqns of $R|U_I$ in $E \times U_I$ are computed as follows:

Let $(x, R) \in E \times U_I$

Notation from last time

$z_1,\dots,z_r$ is a basis of R

$e_1,\dots,e_n$ " of E

$$Z = [z_{ij}]_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \quad ; \quad z_i = \sum z_{ij} e_j$$

$I = \{i_1 < \dots < i_r\}$. Z_I = ~~det~~ submatrix in cols $i_1,\dots,i_r$ of Z.

Z_I is invertible as an element.

$(x, R) \in U_I \iff x \in$ Span of rows of $(Z_I)^{-1} Z$.

Write $x = \sum a_i e_i$

$x = \sum_{u=1}^{r} a_{i_u} w_u$, where w_u is the uth row of Z.

$(x = \sum a_i e_i)$.

The above implies

(*) $a_j = \sum_{t=1}^{r} a_{i_t} z'_{tj}$ $\quad j \notin I$, and the z'_{ij} denote the entries of $(Z_I)^{-1} Z$ in cols in Z.

These define R on $E \times U_I$. Now we need to see that (*) gives a regular sequence on $k[E \times U_I]$ is a polynomial ring of dim $= n + r(n-r)$.

dim $R|_{U_I} = r + r(n-r)$.

Finally (*) has r equations. So it cuts out a regular sequence.

$$\begin{tikzcd}
Z \arrow[d] \arrow[r, hook] \arrow[rr, bend left, "\text{bundle}"] & X \times V \arrow[r, "\varphi"] \arrow[d, "q"] & V \arrow[d] \\
Y \arrow[r, hook] & X \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

$q(Z) = Y$

V projective

$X = \mathbb{A}^n_k$.

Do above for $\mathcal{O}_Z$ & take $q_* K^0(s)$.

$\rightarrow$