

5 Sep 2011

• Motivation: Let k be a field, $X := \text{Space of all } n \times n \text{ matrices}$
 let F and G be f.d. v.s. / k , $\dim F = m$, $\dim G = n$.

• Think $X = \text{Hom}_k(F, G) \cong F^* \otimes_k G$.

$A := k[X]$ coord. ring of X .

A is a polynomial ring: Let $\{f_i\}_{i=1}^m$ & $\{g_j\}_{j=1}^n$ be bases of F and G .

$$\phi_{ij} = f_i \otimes g_j^* \in F \otimes G^*$$

$$A = \text{Sym}(F \otimes G^*) = k[\phi_{ij}, 1 \leq i \leq m, 1 \leq j \leq n]$$

$$\Phi = [\phi_{ij}] . \text{ Let } 0 \leq r \leq \min\{m, n\}.$$

$$Y_r := Y_{2r} := \{ \phi \in X \mid \text{rk } \phi \leq r \}$$

Y is defined by the ideal $\underbrace{I_{r+1}(\Phi)}$

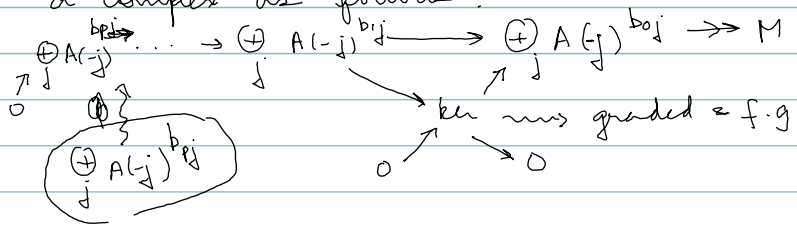
ideal generated by
 the $(r+1) \times (r+1)$ minors.

Aim: To find a (locally) free A -resolution of \mathcal{O}_Y .

Let $V_0 = \text{Rank}$: Free resolution Let A be a polynomial ring over a fld. Think of A as a graded k -alg with generators in degree 1.

$$A = k[x_1, \dots, x_n], \quad x_1, \dots, x_n \text{ indeterminates, } \deg x_i = 1$$

Given any f.g. graded A -module M , we can construct a complex as follows:



Use Hilbert's Syzygy Thm to deduce this stops in finite # of steps

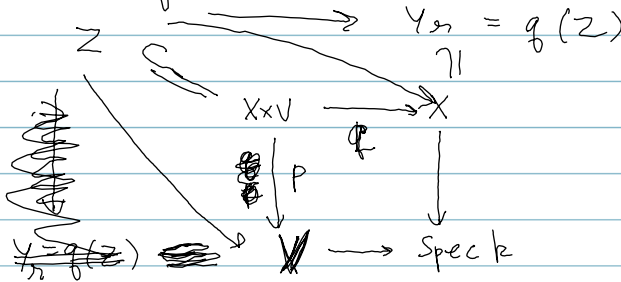
Thus our ~~A-module~~ ^{an} A -resolution of $\mathcal{O}_Y = A/I_{r+1}$ is

$$\begin{array}{ccccccc} & & \text{huge} & & & & \\ & & \rightarrow & A(-r-1) & \rightarrow & A & \rightarrow & A/I_{r+1}(\Phi) \\ \text{Let} & & \times & \text{---} & \times & \text{---} & \times & \end{array}$$

Let $V := \text{Grass}(r, G)$ (= Grassmannian of r -dim subspace of G)

$$X \times V \xrightarrow{\phi} V \quad (X = \text{Spec}(k[\alpha_{ij}]))$$

Think of $X \times V$ as the total space of the trivial vector bundle with fibre $F^* \otimes G$ in G .



Here $Z := \{ (\phi, W) : \text{Im } \phi \in W \}$.

Remark: There exists a Koszul complex on $X \times V$ that gives a locally free $\mathcal{O}_{X \times V}$ -resolution of \mathcal{O}_Z .

Then g_* of this is a complex giving a locally free resolution of \mathcal{O}_Y as an A -module (This is the geometric technique.)

Will now talk about Grassmannians (§3.3).

Let E be a rank n vector space / k .

The Standard Covering of Grass (r, E) : Fix a basis e_1, \dots, e_n of E . Let $R \in \text{Grass}(r, E)$. Let z_1, \dots, z_r be a basis of R . Write

$$z_i = \sum_j z_{ij} e_j \quad 1 \leq i \leq r, \quad 1 \leq j \leq n.$$

Let $1 \leq i_1 < \dots < i_r \leq n$. Define

$$p(i_1, \dots, i_r)(R) := \det \text{ of the } r \times r \text{ submatrix of } [z_{ij}] \text{ consisting of columns } i_1 < \dots < i_r.$$

This defines $p: \text{Grass}(r, E) \rightarrow \mathbb{P}(k^r E)$.

$$R \longmapsto (p(i_1, \dots, i_r)(R)).$$

"Plicker embedding".

Let $I = \{i_1 < \dots < i_r\} \subseteq \{1, \dots, n\}$.

Define $U_I := \{R \in \text{Grass}(r, E) \mid p(i_1, \dots, i_r)(R) \neq 0\}$.

$\{U_I\}_{I \neq \emptyset} : |I|=r, I \subseteq \{1, \dots, n\}$ is an open covering of $\text{Grass}(r, E)$.

Remark: $\forall I, U_I$ is an affine space of dim $r(n-r)$.

After a change of basis, we may assume that $[z_{ij}]$ is such that its columns i_1, \dots, i_r form an $r \times r$ identity matrix. This choice is canonical for any point in U_I .

The remaining $n-r$ (with r rows) give an $r(n-r)$ dim'd affine space.

Tautological bundles :

$$\text{Let } \mathcal{R} = \{ (x, R) \in E \times \text{Grass}(r, E) \mid x \in R \}$$

$p: \mathcal{R} \longrightarrow \text{Grass}(r, E)$, projection on the 2nd coordinate.

This is a rank r sub-bundle of the trivial bundle with fibre E over $\text{Grass}(r, E)$, called the tautological sub-bundle of the trivial bundle E over $\text{Grass}(r, E)$. The quotient is called the tautological quotient bundle.

Proposition: X k scheme, \mathcal{E} a rank r subbundle of the trivial bundle $E \times X$. Then $\exists f: X \longrightarrow \text{Grass}(r, E)$, such that $\mathcal{E} = f^*(\mathcal{R})$.

Sketch: Define $f: X \longrightarrow \text{Grass}(r, E)$ $f^*(\mathcal{R}) = \mathcal{E}$.
 $x \longmapsto E_x$