Cohomology of Vector Bundles and Syzygies Seminar

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• <u>Motivation</u>: Let k be a field, $X :=$ Space of all $n \times n$ matrices

Let F and G be f.d. v.s.$/k$, $\dim F = m$, $\dim G = n$.

• Think $X = \mathrm{Hom}_k(F, G) \simeq F^* \otimes_k G$.

$A := k[X]$ coord. ring of X.

A is a polynomial ring: Let $\{f_i\}_{i=1}^m$ & $\{g_j\}_{j=1}^n$ be bases of F and G.

$$\varphi_{ij} = f_i \otimes g_j^* \in F \otimes G^*$$

$$A = \mathrm{Sym}(F \otimes G^*) = k[\varphi_{ij}, 1 \le i \le m, 1 \le j \le n].$$

$\Phi = [\phi_{ij}]$. Let $0 \le r \le \min\{m, n\}$.

$$Y := Y_r := \{\phi \in X \mid \operatorname{rk} \phi \le r\}$$

Y is defined by the ideal $\underbrace{I_{r+1}(\Phi)}_{\text{ideal generated by the } (r+1)\times(r+1) \text{ minors.}}$

<u>Aim</u>: To find a (locally) free A-resolution of $\mathcal{O}_Y$.

~~Let $V :=$~~ <u>Rmk</u>: <u>Free resolution</u> Let A be a polynomial ring over a fld. Think of A as a graded k-alg with generators in degree 1.

$A = k[x_1, \ldots, x_n]$, $x_1, \ldots, x_n$ indeterminants, $\deg x_i = 1\ \forall$

Given any f.g. graded A-module M, we can construct a complex as follows:

$$\cdots \to \bigoplus_j A(-j)^{b_{1j}} \longrightarrow \bigoplus_j A(-j)^{b_{0j}} \twoheadrightarrow M$$

$\bigoplus_j A(-j)^{b_{pj}} \cdots$; $\ker$ ⟿ graded & f.g. ; $0 \to \ker \to 0$; $\bigoplus_j A(-j)^{b_{pj}}$, 0

Use Hilbert's Syzygy Thm to deduce this stops in finite # of steps

Turns out ~~A-stuff~~ an A-resolution of $O_Y = A/I_{r+1}$ is

$$\to A(-r-1) \xrightarrow{\text{huge}} A \to A/I_{r+1}(\Phi)$$

~~let~~ $\quad * \text{ ——— } * \text{ ——— } *$

Let $V := \mathrm{Grass}(r, G)$ (= Grassmannian of r-dim subspace of G)

$X \times V \xrightarrow{p} V \qquad (X = \mathrm{Spec}\,(k[a_{ij}]))$

Think of $X \times V$ as the total space of the trivial vector bundle with fibre $F^* \otimes G$ on G.

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr, phantom, "\subseteq"] \arrow[drr, bend left] \arrow[ddr] & & Y_r = q(Z) \arrow[d, phantom, "\cap"] \\
& X \times V \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d] \\
& V \arrow[r] & \mathrm{Spec}\, k
\end{tikzcd}$$

Here $Z := \{(\phi, W) : \mathrm{Im}\, \phi \subseteq W\}$.

<u>Remark</u>: There exists a Koszul complex on $X \times V$ that gives a locally free $O_{X\times V}$-resolution of O_Z.

Then q_* of this is a complex giving a locally free resolution of O_Y as an A-module (This is the geometric technique.)

Will now talk about Grassmannians (§3.3).

Let E be a rank n vector space $/k$.

The Standard Covering of Grass(r, E): Fix a basis $e_1, \ldots, e_n$ of E. Let $R \in$ Grass(r, E). Let $z_1, \ldots, z_r$ be a basis of R. Write

$$z_i = \sum z_{ij} e_j \qquad 1 \leq i \leq r, \quad 1 \leq j \leq n.$$

Let $1 \leq i_1 < \cdots < i_r \leq n$. Define

$p(i_1, \ldots, i_r)(R) :=$ det of the $r \times r$ submatrix of $[z_{ij}]$ consisting of columns $i_1 < \ldots < i_r$.

This defines $p:$ Grass$(r, E) \longrightarrow \mathbb{P}(\Lambda^r E)$.

$$R \longmapsto (p(i_1, \ldots, i_r)(R)).$$

"Plücker embedding".

Let $I = \{i_1 < \ldots < i_r\} \subseteq \{1, \ldots, n\}$.

Define $U_I := \{R \in \text{Grass}(r, E) \mid p(i_1, \ldots, i_r)(R) \neq 0\}$.

$\{U_I : |I| = r, I \subseteq \{1, \ldots, n\}\}$ is an open covering of Grass(r, E).

Remark: $\forall I$, U_I is an affine space of dim $r(n-r)$. After a change of basis, we may assume that $[z_{ij}]$ is such that its columns $i_1, \ldots, i_r$ form an $r \times r$ identity matrix. This choice is canonical for any point in U_I .... The remaining $n-r$ (with r rows) give an $r(n-r)$ dim'l affine space.

Tautological bundles:

Let $\mathcal{R} = \{ (x, R) \in E \times \mathrm{Grass}(r, E) \mid x \in R \}$

$\varphi : \mathcal{R} \longrightarrow \mathrm{Grass}(r, E)$, projection on the 2^{nd} coordinate.

This is a rank r sub-bundle of the trivial bundle with fibre E over $\mathrm{Grass}(r, E)$, called the tautological sub-bundle of the trivial bundle E over $\mathrm{Grass}(r, E)$. The quotient is called the tautological quotient bundle.

Proposition: X k scheme. $\mathcal{E}$ a rk r subbundle of the trivial bundle $E \times X$. Then $\exists$ $f : X \longrightarrow \mathrm{Grass}(r, E)$, such that $\mathcal{E} = f^*(\mathcal{R})$.

Sketch: Define $f : X \longrightarrow \mathrm{Grass}(r, E)$, $x \longmapsto \mathcal{E}_x$ $\qquad f^*(\mathcal{R}) = \mathcal{E}$.