Wed 13/07

Algebraic Group Actions assoc. to Equivalence of Sings.

Defn 1) The right group, $\mathcal{R} := \mathrm{Aut}(\mathbb{C}\{x\})$

2) The contact group, $\mathcal{K} := \mathbb{C}\{x\}^* \rtimes \mathcal{R}$

with $(u', \varphi')(u, \varphi) = (u'\varphi'(u), \varphi'\varphi)$

These act on $\mathbb{C}\{x\}$ naturally: ~~as follows~~ :

- $\mathcal{K} \times \mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x\}$

  $((u, \varphi), f) \longmapsto u \cdot \varphi(f)$.

Note that

$$f \overset{r}{\sim} g \Leftrightarrow f \in \mathcal{R}\cdot g \quad ; \quad f \overset{c}{\sim} g \Leftrightarrow f \in \mathcal{K}\cdot g$$

The groups $\mathcal{R}$ and $\mathcal{K}$ are not algebraic as they are infinite dimensional. So we work with their truncations:

$j(\varphi,k)(x_i) = \varphi(x_i)^{(k)}$

$\mathcal{R}^{(k)} := \{\mathrm{jet}(\varphi, k) \mid \varphi \in \mathcal{R}\}$, $\mathcal{K}^{(k)} := \{(\mathrm{jet}(u,k), \mathrm{jet}(\varphi,k)) \mid (u,\varphi) \in \mathcal{K}\}$

These act on the jet space $J^{(k)}$:

$$\varphi \cdot f = \varphi(f)^{(k)} \quad , \quad (u, \varphi)\cdot f = \mathrm{jet}(u\cdot\varphi(f), k)$$

If k is at least as large as the determinacy of g, then $f \overset{r}{\sim} g$ / $f \overset{c}{\sim} g$ iff $g \in \mathcal{R}^{(k)} f$ / $g \in \mathcal{K}^{(k)} f$

Prop. $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are algebraic groups for any $k \geq 1$.

Pf: Any $\varphi \in \mathcal{R}^{(k)}$ is uniquely det'd by

$$\varphi^{(i)} := \varphi(x_i) = \sum_{j=1}^{n} a_j^{(i)} x_j + \sum_{|\alpha|=2}^{k} a_\alpha^{(i)} x^\alpha, \quad i = 1, \dots, n$$

s.t. $\det(a_j^{(i)}) \neq 0$. $\Rightarrow \mathcal{R}^{(k)}$ is an open in a f.d.v.s. It is affine being the complement of a hyp.

Likewise, $\mathcal{K}^{(k)} = \{ (u, \varphi) \mid \varphi \in \mathcal{R}^{(k)},\ u = u_0 + \sum_{|\alpha|=1}^{k} u_\alpha x^\alpha$ with $u_0 \neq 0 \}$

an affine open in an affine space.

The coeffts of $\varphi \cdot \psi$ are poly's in the coeffts of φ, ψ.

Ex: The coeffts of φ^{-1} are polynomials in the coeffts of φ and $1/\det(a_j^{(i)})$. //

Prop'n Let G ~~be~~ $\mathcal{R}^{(k)}$ or $\mathcal{K}^{(k)}$. For $f \in J^{(k)}$, consider the orbit $G \cdot f \subseteq J^{(k)}$; its tangent space at f, $T_f(Gf)$ may be consider a linear subspace of $J^{(k)}$. Then, for $k \geq 1$,

$$T_f(\mathcal{R}^{(k)} f) = (m \cdot j(f) + m^{k+1}) / m^{k+1}$$

$$T_f(\mathcal{K}^{(k)} f) = (m \cdot j(f) + \langle f \rangle + m^{k+1}) / m^{k+1}$$

Pf Have a comm. diagram:

$$\begin{array}{ccc} T_e G & \longrightarrow & T_f(Gf) \\ \cong \downarrow & & \downarrow \cong \\ T_g G & \longrightarrow & T_{gf}(Gf) \end{array}$$

We note that $T_g G \to T_{gf}(Gf)$ is surjective for g generic $\Rightarrow$ $T_e G \to T_f(Gf)$ is surjective

Consider $G = \mathcal{K}^{(k)}$. The other case is similar.

Let $t \longmapsto (u_t, \Phi_t) \in \mathcal{K}^{(k)}$ be a curve st $u_0 = 1$, $\Phi_0 = id$:

$$\Phi(x,t) = x + \varepsilon(x,t) : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to (\mathbb{C}^n, 0)$$

$$u(x,t) = 1 + \delta(x,t) : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to \mathbb{C}$$

with $\varepsilon(x,t) = \varepsilon^1(x)t + \varepsilon^2(x)t^2 + \ldots$, $\varepsilon^i = (\varepsilon^i_1, \ldots, \varepsilon^i_n)$, $\varepsilon^i_j \in m$

and $\delta(x,t) = \delta_1(x)t + \delta_2(x)t^2 + \ldots$, $\delta_i(x) \in \mathbb{C}\{x\}$.

The image of $T_e K^{(k)}$ are all vectors

$$\frac{\partial}{\partial t}\left(1 + \delta(x,t) \cdot f(x + \varepsilon(x,t))\right)\Big|_{t=0} \bmod m^{k+1}$$

$$= \delta_1(x,t) f(x) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x) \cdot \varepsilon^1_j(x) \bmod m^{k+1}$$

//

<u>Cor</u> For $f \in \mathbb{C}\{x\}$, $f(0) = 0$, TFAE :

(a) f has a isolated critical pt.

(b) f is rt. ~~det'd~~ finitely-det'd

(c) f is contact finitely-det'd

<u>Pf</u> (a) $\Rightarrow$ (b) : Proved yesterday.

(b) $\Rightarrow$ (c) : Clear.

(c) $\Rightarrow$ (a) : Let f be contact k-det'd, $g \in m^{k+1}$.

Then, $f_t = f + tg \in \mathcal{K}^{(k+1)} f \bmod m^{k+2}$

$$\Rightarrow g = \frac{\partial f_t}{\partial t}\Big|_{t=0} \in m \cdot j(f) + \langle f \rangle \bmod^{k+2}$$

__Ex:__ $\mathfrak{m}^{k+1} \subset \mathfrak{m} \cdot j(f) + \langle f \rangle$

$\text{RHS} \subseteq j(f) + (f) \Rightarrow \tau(f) < \infty$

and so f has iso. crit.

__Lemma__ Let $f \in \mathfrak{m}^2 \subseteq \mathbb{C}\{x\}$ be an isolated sing.
Let k be st. $\mathfrak{m}^{k+1} \subseteq \mathfrak{m} \cdot j(f)$, resp. $\mathfrak{m}^{k+1} \subseteq \mathfrak{m} \cdot j(f) + \langle f \rangle$.

Set $r\text{-codim}(f) :=$ codim of $\mathcal{R}^{(k)} f$ in $J^{(k)}$

$c\text{-codim}(f) :=$ codim of $\mathcal{K}^{(k)} f$ in $J^{(k)}$

Then, $r\text{-codim}(f) = \mu + n$, $c\text{-codim}(f) = \tau + n$

__Pf__ We'll consider $r\text{-codim}(f)$.

By Thm, $r\text{-codim}(f) = \dim_{\mathbb{C}} \mathbb{C}\{x\} / \mathfrak{m} j(f)$

Have s.e.s.

$$0 \to j(f)/\mathfrak{m} j(f) \longrightarrow \mathbb{C}\{x\}/\mathfrak{m} j(f) \longrightarrow \mathbb{C}\{x\}/j(f) \to 0$$

$\dim_{\mathbb{C}} j(f)/\mathfrak{m} j(f) =$ min. # of gens. of $j(f)$

As $j(f)$ is $\mathfrak{m}$-primary, this # $\geq n$.

But $\left\{ \frac{\partial f}{\partial x_i} \right\}_{i=1}^{n}$ is a generating set.

$$\Rightarrow \dim_{\mathbb{C}} j(f) / \mathfrak{m} j(f) = n$$

The statement follows from ses. //

Simple Singularities

Consider the projections $\mathbb{C}\{x\} \to J^{(k)}$, $k \geq 0$. The preimage of open sets in $J^{(k)}$, $k \geq 0$, generate the coarsest topology on $\mathbb{C}\{x\}$ for which all these projections are continuous ....

<u>Defn</u> A p.s. $f \in \mathbb{C}\{x\}$ is stb. <u>right simple</u>, resp. <u>contact simple</u>, if there exists a nbhd. $U \ni f$ in $\mathbb{C}\{x\}$ s.t. U intersects only finitely many orbits of $\mathcal{R}$, resp. $\mathcal{K}$.

~~Rmk~~ : <u>Propn</u> Let $f \in \mathfrak{m} \subseteq \mathbb{C}\{x\}$ have an isolated sing. Then, $\exists\ U \underset{\text{open}}{\subseteq} \mathbb{C}\{x\}$, $f \in U$, s.t. $\forall\ g \in U$ is right $(\mu(f)+1)$ - determined, resp. contact $(\tau(f)+1)$ - determined.

<u>Pf</u> : Semicontinuity. Exercise!

<u>Cor</u> Let $f \in \mathfrak{m}$ have an isolated sing, and let $k \geq \mu(f)+1$, resp. $k \geq \tau(f)+1$. Then, f is right simple, resp. contact simple, iff there is a nbhd. of $f^{(k)}$ in $J^{(k)}$ which meets only finitely many $\mathcal{R}^{(k)}$- orbits, resp. $\mathcal{K}^{(k)}$- orbits.

$(f \in m^2)$

Goal: Right simple $\Leftrightarrow$ Contact simple

$\Leftrightarrow$ ADE sings.

A_k : $x_1^{k+1} + x_2^2 + \cdots + x_n^2$, $k \geqslant 1$

D_k : $x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \cdots + x_n^2$, $k \geqslant 4$

E_6 : $x_1^3 + x_2^4 + x_3^2 + \cdots + x_4^2$

E_7 : $x_1(x_1^2 + x_2^3) + x_3^2 + \cdots + x_n^2$

E_8 : $x_1^3 + x_2^5 + x_3^2 + \cdots + x_n^2$