Homological Invariants of Monomial and Binomial Ideals

By

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Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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Abstract

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August 2008.

In this dissertation, we study numerical invariants of minimal graded free resolutions of homogeneous ideals in a polynomial ring R. Chapters 2, 3 and 4 deal with homological invariants of edge ideals of bipartite graphs. First, in Chapter 2, we relate regularity and depth of bipartite edge ideals to combinatorial invariants of the graphs. Chapter 3 discusses arithmetic rank, and shows that some classes of Cohen-Macaulay bipartite edge ideals define set-theoretic complete intersections. It is known, due to G. Lyubeznik, that arithmetic rank of a square-free monomial ideal I is at least the projective dimension of R/I. As an application of the results in Chapter 2, we show in Chapter 4 that the multiplicity conjectures of J. Herzog, C. Huneke and H. Srinivasan hold for bipartite edge ideals, and that if the conjectured bounds hold with equality, then the ideals are Cohen-Macaulay and has a pure resolution. Chapter 5 describes joint work with G. Caviglia, showing that any upper bound for projective dimension of an ideal supported on N monomials counted with multiplicity is at least $2^{N/2}$. We give the example of a binomial ideal, whose projective dimension grows exponentially with respect to the number of monomials appearing in a set of generators. Finally, in Chapter 6, we study Alexander duality, giving an alternate proof of a theorem of K. Yanagawa which states that for a square-free monomial ideal I, R/I has Serre's property (S_i) if and only if its Alexander dual has a linear resolution up to homological

degree i. Further, if R/I has property (S_2), then it is locally connected in codimension 1.

"... So you've won the Scripture-knowledge prize, have you?"

"Sir, yes, sir."

"Yes," said Gussie, "you look just the sort of little tick who would. And yet," he said, pausing and eyeing the child keenly, "how are we to know that this has all been open and above board? Let me test you, G. G. Simmons. What was What's-His-Name—the chap who begat Thingummy? Can you answer me that, Simmons?"

"Sir, no, sir."

Gussie turned to the bearded bloke.

"Fishy," he said. "Very fishy. This boy appears to be totally lacking in Scripture knowledge."

The bearded bloke passed a hand across his forehead.

— P. G. Wodehouse, *Right Ho, Jeeves*

Acknowledgments

To my advisor, Craig Huneke, I owe much, for teaching me commutative algebra and his guidance during the work that lead to this thesis. It was always a joy to talk to him, not just about doing mathematics and about its professional aspects, but also about Wodehouse. His dedication to students and the care that he shows for good teaching will be a model for my academic life.

The thesis committee consisted, additionally, of Perry Alexander, Margaret Bayer, Daniel Katz and Jeremy Martin. I thank for them for their comments on the form and the content of the thesis, and, also, for explaining many aspects of commutative algebra and combinatorics throughout my graduate school years. I also express my gratitude to Giulio Caviglia, Hema Srinivasan and Bernd Ulrich who helped me — apart from Craig Huneke and Daniel Katz — to find a post-doctoral position. A part of this thesis is based on joint work with Giulio Caviglia; I thank him for permission to include it.

I have benefited from discussions with many people in commutative algebra and algebraic geometry. Janet Striuli helped me through my first steps in commutative algebra. Also valuable was everything that I learnt from Bangere Purnaprajna. I would like to thank David Eisenbud, Srikanth Iyengar, Gennady Lyubeznik, Ezra Miller, Isabella Novik, Anurag Singh and Irena Swanson for sharing some of their ideas and their encouragement.

A rather happy ride through graduate school that I have had owes itself to many people: Garrin Kimmell and Morgan Shortle, Susana Bernad, Andrew Danby, Mirco Speretta, Andy Hom and Halle O'Neil, Nicole Abaid, Jenny Buontempo, Erin Carmody, Mrinal Das, Aslihan Demirkaya, Tom Enkosky, William Espenschied, Ananth Hariharan, Brandon Humpert, Jeff Mermin, Branden Stone, and Javid Validashti. The support from the Mathematics Department was crucial at every stage.

None of this would have been possible without the encouragement and support of my parents.

Contents

Introduction

Below we present a brief introduction to the problems studied in this dissertation, followed by main results of each chapter. Detailed discussion of the problems and earlier work is given in each chapter.

Main Results

The general motif of this dissertation is the study of numerical invariants of free resolutions over polynomials rings. Let $\Bbbk$ be a field and V a finite set of indeterminates. Let $R = \Bbbk[V]$ and M a finitely generated graded R-module. A graded free resolution of of M is a complex

$$(\mathbb{F}_\bullet, \phi_\bullet): \qquad \cdots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

of graded free R-modules F_l such that the homology groups $\mathrm{H}_0(\mathbb{F}_\bullet) \simeq M$ and $\mathrm{H}_l(\mathbb{F}_\bullet) = 0$ for $l > 0$.

A discussion of graded free resolutions appears in Chapter 1. We review free resolutions of monomial ideals in detail, discussing initial ideals, polarization, and Stanley-Reisner theory. In the next three chapters, we look at free resolutions of edge ideals of bipartite graphs. These are ideals generated by quadratic square-free monomials that could be thought of as edges of a bipartite graph. We study ideals corresponding to perfectly matched bipartite graphs; this class contains the class of unmixed (and Cohen-Macaulay) ideals.

The tool that we use to study the edge ideal of a perfectly matched bipartite graph G is a directed graph $\mathfrak{D}_G$ that we associate to G; see Discussion 2.2.1. We then reformulate some known results in this framework and give different proofs. For example, in Chapter 2, we give an alternate proof of a result of J. Herzog and T. Hibi characterizing Cohen-Macaulay bipartite graphs.

Theorem 2.2.13. *Let G be a bipartite graph on the vertex set $V = V_1 \bigsqcup V_2$. Then G is Cohen-Macaulay if and only if G is perfectly matched and the associated directed graph $\mathfrak{D}_G$ is acyclic and transitively closed, i.e., it is a poset.*

We then proceed to study (Castelnuovo-Mumford) regularity and depth of such ideals. We show that

Theorem 2.2.15. *Let G be an unmixed bipartite graph with edge ideal I. Then* $\operatorname{reg} R/I = \max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_G}\}$. *In particular,* $\operatorname{reg} R/I = r(I)$.

Here $\mathscr{A}_{\mathfrak{D}_G}$ is the set of antichains in $\mathfrak{D}_G$ and $r(I)$ is the maximum size of a pairwise disconnected set of edges. This theorem complements a result of X. Zheng [Zhe04, Theorem 2.18] that if I is the edge ideal of a tree (an acyclic graph) then $\operatorname{reg} R/I = r(I)$. In Theorem 2.2.17, we describe the resolution of the Alexander dual of a Cohen-Macaulay bipartite edge ideal in terms of the antichains of $\mathfrak{D}_G$. We use this, along with a result of N. Terai [Ter99] (see [MS05, Theorem 5.59] also), to give a description of the depth of unmixed bipartite edge ideals in Corollary 2.2.18. We conclude Chapter 2 by describing the Cohen-Macaulay bipartite graphs that have quasi-pure resolutions in Proposition 2.4.5; see the opening paragraph in Section 2.4 also. Quasi-pure resolutions are defined on p. 23. The notions of pure and quasi-pure resolutions have appeared in the the multiplicity conjectures of Herzog, C. Huneke and H. Srinivasan, which we will discuss later.

Chapter 3 is devoted to the study of arithmetic rank of edge ideals of Cohen-Macaulay bipartite graphs; we show that for a certain class of bipartite graphs, the edge ideals define set-theoretic complete intersections.

Theorem 3.2.1. *Let G be a Cohen-Macaulay bipartite graph with edge ideal I. If $\mathfrak{d}_G$ has an embedding in $\mathbb{N}^2$, then* $\operatorname{ara} I = \operatorname{ht} I$.

In Chapter 4, we prove the multiplicity conjectures of Herzog, Huneke and Srinivasan for edge ideals of bipartite graphs. Let $\overline{m}_l$ be the maximum twist appearing at the homological degree l in a minimal graded free resolution of R/I. Let $c = \operatorname{ht} I$. Denote the Hilbert-Samuel multiplicity of R/I by $e(R/I)$. Then

Theorem 4.1.2. *Let $I \subseteq R$ be the edge ideal of a bipartite graph G. Then*

$$e(R/I) \leq \frac{\overline{m}_1 \overline{m}_2 \cdots \overline{m}_c}{c!}.$$

It was further conjectured by J. Migliore, U. Nagel and T. Römer that if equality holds for an ideal I, then R/I is Cohen-Macaulay and has a pure resolution. See p. 23 for the definition of pure resolutions. We prove this conjecture for bipartite edge ideals.

Theorem 4.1.3. *Let I be the edge ideal of a bipartite graph G. If equality holds in Conjecture* (HHSu)*, then R/I is a complete intersection, or is Cohen-Macaulay with* $\operatorname{reg} R/I = 1$. *In either of the cases, R/I is Cohen-Macaulay and has a pure resolution.*

Conjecture (HHSu), mentioned in Theorem 4.1.3, is the assertion that the conclusion of Theorem 4.1.2 is true. Additionally, we prove that the weaker 'Taylor bound' conjecture of Herzog and Srinivasan for monomial ideals is true for all quadratic square-free monomial ideals. Let τ_l be the maximum twist at homological degree l in the Taylor resolution of R/I.

Theorem 4.1.1. *Let $I \subseteq R$ be generated by monomials of degree 2. Then*

$$e(R/I) \leq \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

In Chapter 5, we look at the relation between the size of a monomial support of a homogeneous ideal $I \subseteq R$ and the projective dimension of R/I. This is joint work with G. Caviglia [CK08]. The main result of that chapter is that any bound based on the size of the monomial support must be at least exponential:

Theorem 5.2.3. *Any upper bound for projective dimension of an ideal supported on N monomials counted with multiplicity is at least $2^{N/2}$.*

This gives a partial answer a question raised by Huneke on how good an estimate of projective dimension the size of a monomial support is.

Finally, in Chapter 6, we relate Serre's property (S_i) of a square-free monomial ideal I to linearity of the resolution of its Alexander dual $I^\star$. This was first proved by K. Yanagawa, but the proof we give is different and is built around describing the non-(S_i)-loci of square-free monomial ideals.

Theorem 6.1.2 ([Yan00b, Corollary 3.7])**.** *Let $I \subseteq R$ be a square-free monomial ideal with $\operatorname{ht} I = c$. Then for $i > 1$, the following are equivalent:*

a. R/I satisfies property (S_i).

b. The Alexander dual $I^\star$ satisfies $(N_{c,i})$.

It is known [Har62, Corollary 2.4] that for any ideal I, not necessarily homogeneous, if R/I satisfies property (S_2), then $\operatorname{Spec} R/I$ is locally connected in codimension

1. For square-free monomial ideals, we prove the converse, giving the following equivalence:

Theorem 6.1.5. *Let $R = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\operatorname{Spec} R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2).*

Each chapter concludes with a discussion of future problems and questions.

Notational Conventions

We will work over a field $\Bbbk$ of arbitrary characteristic. We set V to be a finite set of indeterminates over $\Bbbk$. As usual, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers (starting at 0), and the set of integers, respectively. The set of functions from V to $\mathbb{N}$ will be denoted $\mathbb{N}^V$. For any integer $n \geq 1$, let $[n] := \{1, \ldots, n\}$ For a module M, $\lambda(M)$ is its length, which is defined to be the length of any (equivalently, all) composition series as Abelian groups.

For any cardinal n, $n \leq \infty$ means that n is finite; to mean that n is an infinite cardinal, we will write $n = \infty$.

Chapter 1

Graded Free Resolutions

Let $\Bbbk$ be a field and V a finite set of indeterminates. Let $R = \Bbbk[V]$ be the polynomial ring with variables V. Denote the (maximal) ideal generated by all the $x \in V$ by $\mathfrak{m}$. In this thesis, we will need two gradings on R and R-modules, which we now discuss. The general references for this chapter are [Eis95, BH93].

1.1 Graded Modules

For $i \in \mathbb{N}$, denote by R_i the $\Bbbk$-vector space of homogeneous polynomials of degree i. This is a finite dimensional vector space over $\Bbbk$. Given any polynomial in R, we can write it uniquely as a sum of its homogeneous components; hence, as a $\Bbbk$-vector space, $R \simeq \bigoplus_{i\in\mathbb{N}} R_i$. Notice that for all $x \in V$, $\deg x = 1$, so R is generated by R_1 as a $\Bbbk$-algebra. We will refer to this as the *standard grading* of R.

Let M be an R-module. We say that M is *graded* if as a $\Bbbk$-vector space, M has a decomposition $M \simeq \bigoplus_{j\in\mathbb{Z}} M_j$, such that for all $i \in \mathbb{N}$ and for all $j \in \mathbb{Z}$, $R_iM_j \subseteq M_{i+j}$. Analogous to the terminology in the case of R, we say that an element of M_j is *homogeneous* of degree j. Notice that R is a graded module over itself, and is free of rank one. We write $R(-j)$ for a free graded R-module of rank one, with a homogeneous generator of degree j. The degree j of the homogeneous generator is called the *twist* ; we will also refer to $R(-j)$ as R *twisted* j *times*. For each $i \in \mathbb{Z}$, $R(-j)_i = R_{i-j}$, as $\Bbbk$-vector spaces. More generally, if F is a finitely generated free graded R-module of rank b, then there exists $j_1, \ldots, j_b \in \mathbb{Z}$ such that $F \simeq R(-j_1) \oplus \ldots \oplus R(-j_b)$. When an ideal I is generated by homogeneous elements, then we say that I is *homogeneous*.

Another grading for R, which is finer than the standard grading is multigrading. Let $\mathbf{e}_x \in \mathbb{N}^V$ be the function that sends $x \mapsto 1$ and $y \mapsto 0$ for all $y \neq x$, for all $x, y \in V$. We treat R as $\mathbb{N}^V$-graded, by setting, for all $x \in V$, $\deg x = \mathbf{e}_x$. A *monomial* is a polynomial of the form $\prod_{x \in V} x^{\sigma(x)}$, for some $\sigma \in \mathbb{N}^V$. For any function $\sigma \in \mathbb{N}^V$, the set of polynomials $f \in R$ with $\deg f = \sigma$ is a one-dimensional $\Bbbk$-vector space, spanned by the monomial $\prod\limits_{x \in V} x^{\sigma(x)}$. We will call this grading of R *multigrading*. In order to define multigrading for R-modules, let $\mathbb{Z}^V$ be the set of functions from V to $\mathbb{Z}$. For an R-module M, we say that it *multigraded* if we can write $M \simeq \bigoplus\limits_{\sigma' \in \mathbb{Z}^V} M_{\sigma'}$ as $\Bbbk$-vector spaces, such that for all $\sigma \in \mathbb{N}^V$ and $\sigma' \in \mathbb{Z}^V$, $R_\sigma M_{\sigma'} \subseteq M_{\sigma+\sigma'}$. Here we treat $\mathbb{N}^V$ as a subset of $\mathbb{Z}^V$, extending the natural inclusion $\mathbb{N} \subseteq \mathbb{Z}$. An ideal $I \subseteq R$ is said to be a *monomial ideal* if it has a minimal set of generators consisting only of monomials. If $I \subseteq R$ is a monomial ideal, then the minimal generators of I, then its minimal monomial generators are uniquely determined; for, in each $\sigma \in \mathbb{N}^V$, $I_\sigma \subseteq R_\sigma$, and R_σ is a 1-dimensional $\Bbbk$-vector space.

A *multidegree* σ is an element of $\mathbb{Z}^V$. For any multidegree σ, we say that an element of M_σ is *homogeneous* with multidegree σ. We will represent the multidegree σ by the monomial $\prod\limits_{x \in V} x^{\sigma(x)}$. Moreover, if σ is *square-free*, *i.e.*, if $\sigma(x) \in \{0, 1\}$ for all $x \in V$, we will use the set $\{x \in V : \sigma(x) = 1\}$ to represent σ. A free module of rank one, with generator in multidegree $\sigma \in \mathbb{Z}^V$ will be denoted as $R(-\sigma)$, even when we write σ as a monomial, or — in the square-free case — as a subset of V.

We observe that the multigrading is finer than the standard grading, in the sense that multigraded modules are graded in the standard grading. To see this, first, for a multidegree $\sigma \in \mathbb{Z}^V$, set $|\sigma| := \sum_{x \in V} \sigma(x)$. Now let M be any multigraded R-module. For any $j \in \mathbb{Z}$, the define $M_j = \bigoplus_{|\sigma|=j} M_\sigma$ as $\Bbbk$-vector spaces. It is easy to see that $M \simeq \bigoplus_{j \in \mathbb{Z}} M_j$ is a graded R-module in the standard grading. Hence, we will, hereafter, often only say that M is graded; the grading will be clear from the context. The ideal $\mathfrak{m}$ is called the *homogeneous maximal ideal* of R. It is graded in both the standard grading and in the multigrading. It is true that if $N \subseteq M$ are graded modules, then so is M/N. Consequently, $\Bbbk \simeq R/\mathfrak{m}$ is a graded R-module.

Remark 1.1.1. When a multidegree σ is square-free, then $|\sigma| = |\{x \in V : \sigma(x) = 1\}|$. This is consistent with our notation for square-free multidegrees and the common usage of $|\cdot|$ to mean cardinality of sets, which too we will follow. In Chapter 4, we will use $|\cdot|$ for the underlying undirected graph of a directed graph.

For an R-module M, not necessarily graded, we say that a prime ideal $\mathfrak{p} \subseteq R$ is *associated* to M if there is an R-linear homomorphism $R/\mathfrak{p} \hookrightarrow M$, or equivalently, there exists $x \in M$ such that $(0 :_R x) = \mathfrak{p}$. (Here $(0 :_R x) := \{a \in R : ax = 0\}$. Similarly, for an ideal $I \in R$, we may define $(0 :_M I) := \{x \in M : ax = 0 \text{ for all } a \in I\}$, and for $a \in R$, $(0 :_M a) := (0 :_M (a))$.) The set of associated primes of M will be denoted by $\operatorname{Ass} M$, and is a finite set. By $\operatorname{Unm} M$, we denote the set of *unmixed* associated primes, *i.e.*, those $\mathfrak{p} \in \operatorname{Ass} M$ such that $\dim R/\mathfrak{p} = \dim M$. The *socle* of M is $\operatorname{soc} M := (0 :_M \mathfrak{m})$. If M is a standard graded module, its associated primes are homogeneous in the standard grading of R. Similarly, if M is multigraded, the associated primes are multigraded. We remark that multigraded prime ideals are generated by subsets of V.

Let M be a finitely generated graded module. We say that a sequence of homogeneous elements $a_1, \ldots, a_t \in R$ is a *regular sequence* on M if $M \neq (a_1, \ldots, a_t)M$, $(0 :_M a_1) = 0$, and for all $1 \leq i \leq t-1$, $((a_1, \ldots, a_i) :_M a_{i+1}) = (a_1, \ldots, a_i)M$. Every maximal regular sequence on M has the same length, called the *depth* of M, and denoted

$\operatorname{depth}_R M$. It is known that $\operatorname{depth}_R M \le \dim M$; see, *e.g.*, [BH93, Proposition 1.2.12]. We say that M is *Cohen-Macaulay* if $\operatorname{depth}_R M = \dim M$.

Proposition 1.1.2. *Let M be a finitely generated module. Then $\bigcup_{x \in M} (0 :_R x) = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$. In particular, if M is graded, then $\operatorname{depth}_R M = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass} M$, i.e., if and only if $\operatorname{soc} M \neq 0$.*

1.2 Graded Free Resolutions and Betti Numbers

Let M be an R-module. A *free resolution* of M is a complex

$$(\mathbb{F}_\bullet, \phi_\bullet): \qquad \cdots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

of free R-modules F_l such that the homology groups $\mathrm{H}_0(\mathbb{F}_\bullet) \simeq M$ and $\mathrm{H}_l(\mathbb{F}_\bullet) = 0$ for $l > 0$. We often write $\mathbb{F}_\bullet$, without explicit reference to the maps ϕ_l. When M is finitely generated, we may take the F_l to be of finite rank. We say that $\mathbb{F}_\bullet$ is *minimal* if, for all $l > 0$, $\phi_l(F_l) \subseteq \mathfrak{m} F_{l-1}$. In addition, if M is graded (either in the standard grading, or in the multigrading) then we may take the F_l to be graded too, in the same grading as of M. Moreover, we may assume that all the ϕ_l are homogeneous of degree 0, *i.e.*, for all $l > 1$, and for all $f \in F_l$, $\deg \phi_l(f) = \deg f$. If this happens, we say that $\mathbb{F}_\bullet$ is *graded*. It follows from Nakayama's lemma that every graded R-module has a minimal graded free resolution. Further, when M is finitely generated, the Hilbert syzygy theorem asserts that the minimal graded free resolution of M has finite length, which is at most $|V|$. In other words, there exists $0 \le p \le |V|$ and a complex

$$(\mathbb{F}_\bullet, \phi_\bullet): \qquad 0 \longrightarrow F_p \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

of free graded R-modules of finite rank such that $\mathrm{H}_0(\mathbb{F}_\bullet) \simeq M$ and $\mathrm{H}_l(\mathbb{F}_\bullet) = 0$ for $l > 0$, the maps ϕ_l are homogeneous of degree 0, and $\phi_l(F_l) \subseteq \mathfrak{m}F_{l-1}$.

Let M and N be graded R-modules, with graded free resolutions $\mathbb{F}_\bullet$ and $\mathbb{G}_\bullet$ respectively. Define

$$\mathrm{Tor}_l^R(M,N) := \mathrm{H}_l(\mathbb{F}_\bullet \otimes_R N) \simeq \mathrm{H}_l(\mathbb{G}_\bullet \otimes_R M).$$

The $\mathrm{Tor}_l^R(M,N)$ modules are graded in the same grading as M and N are. Moreover, they are independent of choice of the resolutions of M and N.

We will first define Betti numbers for standard grading. Let M be a standard graded R-module. For $0 \le l \le |V|$ and $j \in \mathbb{Z}$, define *graded Betti numbers*, $\beta_{l,j}(M)$, of M as $\beta_{l,j}(M) := \dim_{\Bbbk} \mathrm{Tor}_l^R(M,\Bbbk)_j$. Similarly, if M is a multigraded R-module, then we define, for $0 \le l \le |V|$ and $\sigma \in \mathbb{Z}^V$, the *multigraded Betti numbers* $\beta_{l,\sigma}(M) := \dim_{\Bbbk} \mathrm{Tor}_l^R(M,\Bbbk)_\sigma$.

Proposition 1.2.1. *Let* $(\mathbb{F}_\bullet,\phi_\bullet) : 0 \longrightarrow F_p \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$ *be a minimal graded free resolution of M. Then for all l and j, the graded free module F_l contains precisely $\beta_{l,j}(M)$ copies of $R(-j)$. If M is multigraded, then, for all $\sigma \in \mathbb{Z}^V$, F_l contains precisely $\beta_{l,\sigma}(M)$ copies of $R(-\sigma)$.*

Proof. The second statement is an easy extension of the first, which we now prove. We can choose homogeneous bases for the F_l, and write $F_l \simeq \bigoplus_j R(-j)^{b_{l,j}}$ for some $b_{l,j} \in \mathbb{N}$. We need to show that $b_{l,j} = \beta_{l,j}(M)$. By definition, for all $l > 0$, $\phi_l(F_l) \subseteq \mathfrak{m}F_{l-1}$. Hence we can express each ϕ_l as a matrix of homogeneous forms all of which belong to $\mathfrak{m}$. Applying $- \otimes_R \Bbbk$, we get

$$\mathbb{F}_\bullet \otimes_R \Bbbk : 0 \longrightarrow \bigoplus_j \Bbbk(-j)^{b_{p,j}} \xrightarrow{0} \cdots \xrightarrow{0} \bigoplus_j \Bbbk(-j)^{b_{1,j}} \xrightarrow{0} \bigoplus_j \Bbbk(-j)^{b_{0,j}} \longrightarrow 0\,,$$

from which we see that $\operatorname{Tor}_l(M,\Bbbk) \simeq \Bbbk(-j)^{b_{l,j}}$. The proposition now follows easily. $\square$

Remark 1.2.2. We observe, as a corollary to the above proposition, that for a multigraded module M, $\beta_{l,j}(M) = \sum_{|\sigma|=j} \beta_{l,\sigma}(M)$. For any homogeneous ideal I, $\beta_{1,j}(R/I)$ is the number of minimal homogeneous generators of I in degree j. Similarly, for a monomial ideal I, $\beta_{1,\sigma}(R/I)$ is the number of minimal generators of I in multidegree σ. Since $\dim_{\Bbbk} R_\sigma = 1$, we see that $\beta_{1,\sigma}(R/I) \leq 1$. Moreover, the minimal monomial generators of I are uniquely defined.

Let M be a finitely generated graded R-module. Two important homological invariants of M that we study in this thesis are projective dimension and regularity. The *projective dimension*, $\operatorname{pd}_R M$, of M is the largest homological degree l such that $\operatorname{Tor}_l^R(M,\Bbbk) \neq 0$. The *(Castelnuovo-Mumford) regularity*, $\operatorname{reg}_R M$, is defined as follows: $\operatorname{reg}_R M := \max\{j-l : 0 \leq l \leq |V|, \operatorname{Tor}_l^R(M,\Bbbk)_j \neq 0\}$. Often, when the ring under consideration is clear from the context, we will only write $\operatorname{pd} M$ and $\operatorname{reg} M$.

Let $(\mathbb{F}_\bullet, \phi_\bullet)$ be a minimal graded free resolution of M. It follows from Proposition 1.2.1 that $F_l = 0$ for $l > \operatorname{pd} M$. Therefore $\operatorname{pd} M$ is the length of a (equivalently, every) minimal graded free resolution of M.

In order to give a similar description of regularity, we consider the case of a finitely generated graded module M generated by elements of non-negative degrees. For example, $M = R/I$ for a homogeneous ideal I. We now introduce some notation for the the minimum and maximum degrees of any minimal generator of F_l.

Definition 1.2.3. *Let M be a finitely generated graded R-module and $0 \leq l \leq \operatorname{pd} M$. The* minimum twist of M at homological degree l *is $\underline{m}_l = \min\{j : \beta_{l,j}(M) \neq 0\}$ The* maximum twist of M at homological degree l *is $\overline{m}_l = \max\{j : \beta_{l,j}(M) \neq 0\}$.*

Lemma 1.2.4. *With notation as above, for $1 \leq l \leq \operatorname{pd} M$, $\underline{m}_l > \underline{m}_{l-1}$. Let $c = \operatorname{ht}(0 :_R M)$. Then for $1 \leq l \leq c$, $\overline{m}_l > \overline{m}_{l-1}$.*

Proof. Let $f \in F_l, f \neq 0$ such that $\deg f = \underline{m}_l$. By minimality of $\mathbb{F}_\bullet$, $\phi_{l+1}(F_{l+1}) = \ker \phi_l \subseteq \mathfrak{m}F_l$. Every non-zero element of $\mathfrak{m}F_l$ has degree greater than $\underline{m}_l$; hence $f \notin \ker \phi_l$. We can therefore write $\phi_l(f) = \sum a_i g_i$, where the g_i form part of a homogeneous R-basis for F_{l-1} and the $a_i \in R$ are non-zero homogeneous forms. By minimality of $\mathbb{F}_\bullet$, we have $a_i \in \mathfrak{m}$, for all i. Hence, for all i, $\underline{m}_{l-1} \leq \deg g_i = \deg \phi_l(f) - \deg a_i < \deg f = \underline{m}_l$.

To prove the second statement, we apply $\operatorname{Hom}_R(-,R)$ to $\mathbb{F}_\bullet$ and observe that

$$0 \longrightarrow \operatorname{Hom}_R(F_0,R) \longrightarrow \operatorname{Hom}_R(F_1,R) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(F_c,R) \longrightarrow 0$$

is a minimal graded free resolution of $\operatorname{Ext}^c_R(M,R)$. We thus obtain that, for $0 \leq l \leq c$, $\underline{m}_l(\operatorname{Ext}^c_R(M,R)) = -\overline{m}_{c-l}(M)$. Applying the first statement to $\underline{m}_l(\operatorname{Ext}^c_R(M,R))$ now concludes the proof. $\square$

Observe that $\operatorname{reg} M = \max\{\overline{m}_l - l : 0 \leq l \leq \operatorname{pd} M\}$. Now assume that M is generated in non-negative degrees. From Lemma 1.2.4, it follows that $\overline{m}_l \geq \underline{m}_l \geq l$ for all l. Hence, for each l, $\overline{m}_l - i$ gives a sense of the "width" of a minimal graded free resolution of M at homological degree i. Therefore $\operatorname{reg} M$ is a heuristic measure of the width of a minimal graded free resolution. In this sense, both projective dimension and regularity as considered as measures of the complexity of a module.

The next proposition relates projective dimension and depth and shows that M is Cohen-Macaulay if and only if $\operatorname{pd} M = \dim R - \dim M$.

Proposition 1.2.5 (Auslander-Buchsbaum formula [Eis95, Theorem 19.9])**.** *Let M be a finitely generated graded module. Then $\operatorname{pd}_R M + \operatorname{depth}_R M = \dim R$.*

Discussion 1.2.6. Let M be a finitely generated graded R-module, with a minimal graded free resolution $\mathbb{F}_\bullet$. The *Hilbert function* of M is the function $\mathfrak{h}_M : \mathbb{Z} \to \mathbb{N}$ taking $i \mapsto \dim_{\Bbbk} M_i$. Let $p = \operatorname{pd}_R M$. Then for all $i \in \mathbb{Z}$, $\dim_{\Bbbk} M_i = \sum\limits_{0 \le l \le p} (-1)^l \dim_{\Bbbk} (F_l)_i$. Since the F_l are finitely generated graded free modules, $\dim_{\Bbbk}(F_l)_i$ is a sum of binomial coefficients, each of which are polynomials of degree $|V|$ in i. Hence there is a polynomial $\pi_M \in \mathbb{Q}[t]$ such that for all $i \gg 0$, $\pi_M(i) = \mathfrak{h}_M(i)$; it is called the *Hilbert polynomial* of M. First assume that M is not a finite-length module. Then $\deg \pi_M = \dim M - 1$ and there exists a positive integer $e(M)$ such that $\pi_M(t) = \frac{e(M)}{(\dim M - 1)!} t^{\dim M - 1} +$ terms of lower degree. If $\lambda(M) < \infty$, then $\pi_M \equiv 0$; we set $e(M) = \lambda(M)$. In either of the cases, we call $e(M)$ the *Hilbert-Samuel multiplicity* of M. Now suppose that $M = R/I$ for some homogeneous ideal I with $\operatorname{ht} I = c$. The following result was first proved by C. Peskine and L. Szpiro [PS74].

$$\sum_{l=0}^{p} (-1)^l \sum_j \beta_{l,j}(R/I) j^t = \begin{cases} 0, & \text{if } 1 \le t \le c-1 \\ (-1)^c c! e(R/I), & \text{if } t = c. \end{cases} \tag{1.1}$$

We say that R/I has a *pure resolution* if for each l, there is a unique twist in the free module $\mathbb{F}_l$, or, equivalently, that $\overline{m}_l = \underline{m}_l$. We say that R/I has a *quasi-pure resolution* if for each l, $\underline{m}_{l+1} \le \overline{m}_l$.

A diagrammatic representation for the graded Betti numbers is the Betti table. Let $\beta_{l,j} = \beta_{l,j}(M), 0 \le i \le \operatorname{pd} M, j \in \mathbb{Z}$ be the graded Betti numbers of a finitely generated graded module M. Then the *Betti table* of M is array with columns indexed by homological degrees l having the entry $\beta_{l,l+j}$ in the row indexed by j. The top-most row gives the *total* Betti numbers, $\beta_l = \sum_j \beta_{l,j}$. This is described in Table 1.1. An example of a Betti table appears in Table 2.1.

	0	1	2	$\cdots$	l	$l+1$	$\cdots$
total	β_0	β_1	β_2	$\cdots$	β_l	β_{l+1}	$\cdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\cdots$	$\beta_{l,l}$	$\beta_{l+1,l+1}$	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{l,l+1}$	$\beta_{l+1,l+2}$	$\cdots$
				$\ddots$			
j	$\beta_{0,j}$	$\beta_{1,1+j}$	$\beta_{2,2+j}$	$\cdots$	$\beta_{l,l+j}$	$\beta_{l+1,l+1+j}$	$\cdots$
j+1	$\beta_{0,j+1}$	$\beta_{1,2+j}$	$\beta_{2,3+j}$	$\cdots$	$\beta_{l,l+j+1}$	$\beta_{l+1,l+2+j}$	$\cdots$

Table 1.1: Betti table

1.3 Monomial Ideals

In this section, we discuss some facts about monomial ideals. First, we describe initial ideals, which are monomial ideals constructed from arbitrary ideals. Then we study Taylor resolutions, which are multigraded free resolutions, often non-minimal, of monomial ideals. We continue with studying polarization, which converts arbitrary monomial ideals to square-free monomial ideals, preserving many numerical characteristics of free resolutions. Finally, we discuss simplicial complexes and Hochster's formula, which relates multigraded Betti numbers of square-free monomials with homology of simplicial complexes.

1.3.1 Initial Ideals

A *monomial order* $>$ on R is a total order on the set of monomials in R, such that, for all monomials f with $f \neq 1$, $f > 1$, and for all monomials $g > g'$, $fg > fg'$. If $f \in R$ is a polynomial, then the *initial term* , $\mathrm{in}_{>} f$, is the largest monomial that appears with a non-zero coefficient in f. Let $I \subseteq R$ be an ideal, not necessarily homogeneous. The *initial ideal*, $\mathrm{in}_{>} I$, of I is the ideal generated by the monomials $\{\mathrm{in}_{>} f : f \in I\}$. The next theorem captures many essential features of initial ideals, and is a standard result.

Theorem 1.3.1. *Let $I \subseteq R$ be a homogeneous ideal and $>$ a monomial order. Then R/I and $R/\operatorname{in}_> I$ have identical Hilbert functions. Moreover, for all l, j, $\beta_{l,j}(R/\operatorname{in}_> I) \geq \beta_{l,j}(R/I)$. In particular,* $\operatorname{ht} I = \operatorname{ht} \operatorname{in}_> I$, $e(R/I) = e(R/\operatorname{in}_> I)$, $\operatorname{pd}(R/\operatorname{in}_> I) \geq \operatorname{pd}(R/I)$ *and* $\operatorname{reg}(R/\operatorname{in}_> I) \geq \operatorname{reg} R/I$.

A *Gröbner basis* for I is a set of polynomials $\{g_1, \ldots, g_s\} \subseteq I$ such that $\operatorname{in}_> I = (\operatorname{in}_> g_1, \ldots, \operatorname{in}_> g_s)$.

1.3.2 The Taylor Resolution

Let $f_1, \ldots, f_m$ be monomials, and let $I = (f_1, \ldots, f_m)$. Then the multigraded module R/I has a multigraded resolution called the *Taylor resolution*. Below we follow the description of [Eis95, Ex. 17.11].

For $S \subseteq [m]$, we set $f_S = \operatorname{lcm}\{f_i : i \in S\}$. At homological degree l, $0 \leq l \leq m$, $T_l := \bigoplus\limits_{|S|=i} R(-f_S)$. This is a multigraded free R-module of rank $\binom{m}{l}$. Here we have used monomials to represent multidegrees. Let $S = \{i_1 < \ldots < i_l\} \subseteq [m]$. For $S' \subseteq [m], |S'| = l-1$, define

$$\varepsilon_{S,S'} = \begin{cases} (-1)^k f_S/f_{S'} & \text{if } S' = S \setminus \{i_k\} \text{ for some } k, \\ 0 & \text{otherwise} \end{cases}$$

Define $\phi_l : T_l \to T_{l-1}$ by sending $f_S \mapsto \sum\limits_{|S'|=l-1} \varepsilon_{S,S'} f_{S'}$. Then

$$\mathbb{T}_\bullet : \qquad 0 \longrightarrow T_m \xrightarrow{\phi_m} \cdots \longrightarrow T_1 \xrightarrow{\phi_1} T_0 \longrightarrow 0$$

is a multigraded free resolution of R/I. Taylor resolutions, in general, are non-minimal. This is clear when $m > |V|$ since $\mathbb{T}_\bullet$ has length m while every finitely generated R-module has a minimal graded free resolution of length at most $|V|$, by the Hilbert syzygy theorem. Here is an example of an ideal generated by three monomials in three variables, with a non-minimal Taylor resolution.

Example 1.3.2. Let $R = \Bbbk[x,y,z]$ and $I = (xy,xz,yz)$. Then the Taylor resolution of R/I is:

$$0 \longrightarrow R(-xyz) \xrightarrow{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}} R(-xyz)^3 \xrightarrow{\begin{bmatrix} z & z & 0 \\ -y & 0 & y \\ 0 & -x & -x \end{bmatrix}} \begin{matrix} R(-xy) \\ \oplus \\ R(-xz) \\ \oplus \\ R(-yz) \end{matrix} \xrightarrow{\begin{bmatrix} xy & xz & yz \end{bmatrix}} R \longrightarrow 0$$

Since $I = (x,y) \cap (x,z) \cap (y,z)$, $\operatorname{ht} I = 2$. Therefore $\dim R/I = 1$. Further, since R/I is reduced, $\operatorname{depth} R/I = 1$. By the Auslander-Buchsbaum formula (Proposition 1.2.5), $\operatorname{pd} R/I = 2$, while the Taylor resolution has length three.

1.3.3 Polarization

Many questions about numerical invariants of graded free resolutions of arbitrary monomial ideals can be reduced to the case of square-free monomial ideals by polarization. Resolutions of square-free monomial ideals are related to simplicial homology, and in the next subsection, we will look at some of these techniques. We follow the description of [MS05, Section 3.2 and Exercise 3.15].

In this section we will label the indeterminates in V as $x_1,\ldots,x_n$. Let $f = x_1^{e_1}\cdots x_n^{e_n}$ be a monomial. Let $\tilde{R} = \Bbbk[x_{11},x_{12},\ldots,x_{21},x_{22},\ldots,x_{n1},x_{n2},\ldots]$. Then a *polarization* of f in $\tilde{R}$ is the square-free monomial $\tilde{f} = x_{11}\cdots x_{1e_1}x_{21}\cdots x_{2e_2}\cdots x_{n1}\cdots x_{ne_n}$. If $f_1,\ldots,f_m \in R$ are monomials and $I = (f_1,\ldots,f_m)$, then we call the square-free monomial ideal $\tilde{I}$ generated by the polarizations of the f_i in a larger polynomial ring $\tilde{R}$, a polarization of I.

Example 1.3.3. Let $R = \Bbbk[x,y,z]$ and $I = (x^3, y^2, x^2z^3)$. Then a polarization of I in $\tilde{R} = \Bbbk[x_1,x_2,x_3,y_1,y_2,z_1,z_2,z_3]$ is the ideal $\tilde{I} = (x_1x_2x_3, y_1y_2, x_1x_2z_1z_2z_3)$.

The next proposition describes some elementary properties of polarization.

Proposition 1.3.4. *Let $I \subseteq R = \Bbbk[x_1,\dots,x_n]$ be a monomial ideal. If $\tilde{I} \subseteq \tilde{R}$ is a polarization of I, then for all l, j, $\beta_{l,j}(R/I) = \beta_{l,j}(\tilde{R}/\tilde{I})$. In particular,* $\operatorname{ht} I = \operatorname{ht} \tilde{I}$ *and $e(R/I) = e(\tilde{R}/\tilde{I})$.*

Proof. That height and multiplicity remain unchanged is an easy consequence of the assertion about graded Betti numbers and the formula of Peskine-Szpiro, given in Equation 1.1. To prove the statement about graded Betti numbers, we follow an argument of A. Taylor [Tay00, Proposition 1.12].

Let $f_1,\dots,f_m$ be the minimal monomial generators of I. We first describe a *partial* polarization I_1 of I, and prove the assertion for I_1. It is easy to see that repeatedly applying this procedure, we can obtain $\tilde{I}$, and that the conclusion holds for $\tilde{I}$.

Let e_1 be the largest exponent with which x_1 appears in the f_i. We may assume, without loss of generality, that $e_1 > 1$. Let $R_1 = \Bbbk[x_1,\dots,x_n,y]$. For $1 \le i \le m$, define $g_i \in R_1$ by

$$g_i = \begin{cases} \frac{f_i}{x_1}y, & \text{if } x_1^{e_1} | f_i, \\ f_i, & \text{otherwise.} \end{cases}$$

To prove the statement about the graded Betti numbers, we make a sequence of claims: that $y - x_1$ is a non-zerodivisor on R_1/I_1 and that $R_1/(I_1, y-x_1) \simeq R/I$.

If, on the contrary, $y - x_1$ were a zerodivisor on R_1/I_1, then there exists $\mathfrak{p} \in \operatorname{Ass} R_1/I_1$ such that $y - x_1 \in \mathfrak{p}$. Since I_1 and $\mathfrak{p}$ are monomial ideals, there exists a monomial $f \notin I_1$ such that $\mathfrak{p} = (I_1 :_{R_1} f)$. Then $f(y - x_1) \in I_1$. Again, since I_1 is a monomial ideal, $fy \in I_1$ and $fx_1 \in I_1$. Let i, be such that $g_i | fy$ and $g_j | fx_1$. If $y \nmid g_i$, then $g_i | f$, which contradicts the fact that $f \notin I_1$; hence $y | g_i$. We can write $g_i = x_1^{e_1-1}yh$, where h is not divisible by x_1 or by y. Hence $x_1^{e_1-1}h|f$. We now claim that $g_j|f$. If not, then the degree of x_1 in g_j is precisely one more than the degree of x_1 in f, as $g_j|fx_1$. That is, $x_1^{e_1}|g_j$, a contradiction as, by definition, e_1-1 is the largest exponent of x_1 dividing any of the minimal monomial generators of I_1.

□

1.3.4 Stanley-Reisner Theory and Alexander Duality

One of the primary advantages of reducing problems about resolutions of arbitrary monomial ideals to the case of square-free monomial ideals is its relation to homology of simplicial complexes, developed by R. Stanley, G. Reisner and M. Hochster. General references for this section are [Hoc77] and [MS05, Chapter 1 and Section 5.1]. Recall that $R = \Bbbk[V]$ is the polynomial ring over a finite set of indeterminates V. Let $I \subseteq R$ be a square-free monomial ideal.

A *abstract simplicial complex* Δ on the set V is a collection of subsets $F \subseteq V$, called *faces*, such that $\{x\} \in \Delta$ for all $x \in V$ and for all $F \in \Delta$ and for all $G \subseteq F$, $G \in \Delta$. Hereafter, we will omit “abstract”, while referring to simplicial complex. The *dimension*, $\dim F$, of a face F is $|F|-1$. The dimension, denoted $\dim \Delta$, of Δ is the maximum among the dimensions of the faces of Δ. A *facet* of Δ is a face that is maximal under inclusion. We call zero- and one-dimensional faces, *vertices* and *edges* respectively. Hence, vertices of Δ are singleton subsets of V; identifying them with elements of V, we will say that V is the *vertex set* of Δ. We will adopt the convention of representing a face F of Δ by a square-free monomial, instead of as a set. In this notation, inclusion of subsets of V corresponds to divisibility of monomials. Moreover, it suffices only to give the maximal faces of Δ; we can construct the other faces by taking subsets.

Example 1.3.5. Let $V = \{a,b,c,d\}$. Then $\Delta = \{\emptyset, a, b, c, d, ab, ac, bc, ad, bd, abc\}$ is a simplicial complex. We write this as $\Delta = \langle abc, ad, bd \rangle$.

Let Δ be a simplicial complex on a (finite, always) vertex set V. The *Stanley-Reisner ideal* of Δ is the ideal generated by the square-free monomials $\prod\limits_{x \in F} x$ where $F \subseteq V$ is such that $F \notin \Delta$. Such an F is often called a *non-face* of Δ. It follows easily that the Stanley-Reisner ideal of Δ is generated by the minimal non-faces of Δ Let I be the Stanley-Reisner ideal of Δ. Then we call R/I the *Stanley-Reisner ring* of Δ. Similarly, if $I \subseteq R = \Bbbk[V]$ is a square-free monomial ideal, then the square-free monomials of $R \setminus I$ form a simplicial complex, called the *Stanley-Reisner complex* of I. These two constructions are inverses of each other, and describe the *Stanley-Reisner correspondence* between simplicial complexes on V and square-free monomial ideals in $\Bbbk[V]$. This correspondence has an equivalent description. Let Δ be a simplicial complex on V and I its Stanley-Reisner ideal. Then $I = \bigcap\limits_{F \in \Delta} (\bar{F})R$ where $(\bar{F})R$ denotes the prime ideal generated by $\bar{F} := V \setminus F$ (see [MS05, Theorem 1.7]). Hence the minimal prime ideals of R/I correspond to complements of maximal faces of Δ. We say that a simplicial complex is *Cohen-Macaulay* (respectively, *Gorenstein*) if its Stanley-Reisner ring is Cohen-Macaulay (respectively, Gorenstein).

Let Δ be a simplicial complex on V. The *Alexander dual* of Δ, denoted $\Delta^\star$, is the simplicial complex $\{F : \bar{F} \notin \Delta\}$ of complements of non-faces of Δ. Let $I \subseteq R = \Bbbk[V]$ be a square-free monomial ideal. Let m and $F_i \subseteq V, 1 \le i \le m$ be such that $\prod_{x \in F_i} x, 1 \le i \le m$ are the minimal monomial generators of I. The *Alexander dual* of I, denoted $I^\star$, is the square-free monomial ideal $\cap_{i=1}^m (F_i)$. If I is the Stanley-Reisner ideal of Δ, then $\bar{F_i}, 1 \le i \le m$ are precisely the facets of $\Delta^\star$. Hence $I^\star$ is the Stanley-Reisner ideal of $\Delta^\star$.The following is an important result on Alexander duality:

Proposition 1.3.6 (Terai [Ter99]; [MS05, Theorem 5.59]). *For any square-free monomial ideal J,* $\operatorname{pd} R/J = \operatorname{reg} J^\star$.

Discussion 1.3.7 (Hochster's Formula [MS05, Corollary 5.12 and Corollary 1.40]). Graded free resolutions of square-free monomials are closely related to the homological properties of simplicial complexes. Let Δ be simplicial complex on V with Stanley-Reisner ideal I. For $\sigma \subseteq V$, we denote by $\Delta|_\sigma$ the simplicial complex obtained by taking all the faces of Δ whose vertices belong to σ. Note that $\Delta|_\sigma$ is the Stanley-Reisner complex of the ideal $I \cap \Bbbk[\sigma]$. Similarly, define the *link*, $\operatorname{lk}_\Delta(\sigma)$, of σ in Δ to be the simplicial complex $\{F \setminus \sigma : F \in \Delta, \sigma \subseteq F\}$. Its Stanley-Reisner ideal in $\Bbbk[\bar{\sigma}]$ is $(I : \sigma) \cap \Bbbk[\bar{\sigma}]$. Multidegrees σ with $\beta_{l,\sigma}(R/I) \neq 0$ are square-free. Further $\beta_{l,\sigma}(R/I) = \dim_\Bbbk \widetilde{\mathrm{H}}_{|\sigma|-l-1}(\Delta|_\sigma; \Bbbk)$ and $\beta_{l,\sigma}(I^\star) = \dim_\Bbbk \widetilde{\mathrm{H}}_{l-1}(\operatorname{lk}_\Delta(\bar{\sigma}); \Bbbk)$. Relating these two formulas, we have that

$$\dim \widetilde{\mathrm{H}}_{l-1}(\operatorname{lk}_\Delta(\bar{\sigma}); \Bbbk) = \beta^{\Bbbk[\sigma]}_{|\sigma|-l,\sigma}\left(\frac{\Bbbk[\sigma]}{(I : \bar{\sigma}) \cap \Bbbk[\sigma]}\right) = \beta^{R}_{|\sigma|-l,\sigma}\left(\frac{R}{(I : \bar{\sigma})}\right).$$

(That the map $\Bbbk[\sigma] \to R$ is faithfully flat gives the second equality.) We summarize this below for later use:

$$\beta_{l,\sigma}(I^\star) = \beta_{|\sigma|-l,\sigma}\left(\frac{R}{(I : \bar{\sigma})}\right). \tag{1.2}$$

We add, parenthetically, that links of faces in Cohen-Macaulay complexes are themselves Cohen-Macaulay. □

Now assume that I is a square-free monomial ideal. Let Δ be the Stanley-Reisner complex of I. We now describe how the graded Betti numbers change under restriction to a subset of the variables and under taking colons.

Lemma 1.3.8. *Let $I \subseteq R = \Bbbk[V]$ be a square-free monomial ideal, $x \in V$, $l, j \in \mathbb{N}$ and $\sigma \subseteq V$ with $|\sigma| = j$. Then* *a.* *Let $W \subseteq V$ and $J = (I \cap \Bbbk[W])R$. Then,*

$$\beta_{l,\sigma}(R/J) = \begin{cases} 0, & \sigma \not\subseteq W, \\ \beta_{l,\sigma}(R/I), & \sigma \subseteq W. \end{cases}$$

In particular, $\beta_{l,j}(R/J) \leq \beta_{l,j}(R/I)$.

b. *If $\beta_{l,\sigma}(R/(I:x)) \neq 0$, then $\beta_{l,\sigma}(R/I) \neq 0$ or $\beta_{l,\sigma\cup\{x\}}(R/I) \neq 0$.*

Proof. (a): The second assertion follows from the first, which we now prove. Let $\tilde{\Delta}$ be the Stanley-Reisner complex of J. Since for all $x \in V \setminus W$, x does not belong to any minimal prime ideal of R/J, we see that every maximal face of $\tilde{\Delta}$ is contains $V \setminus W$. Hence if $\sigma \not\subseteq W$, then for all $x \in \sigma \setminus W$, $\tilde{\Delta}|_\sigma$ is a cone with vertex x, which, being contractible, does not have any homology. Applying Hochster's formula (Discussion 1.3.7) we see that $\beta_{l,\sigma}(R/J) = 0$.

Now let $\sigma \subseteq W$ and $F \subseteq V$. Then $F \in \Delta|_\sigma$ if and only if $I \subseteq (\bar{F})R$ and $F \subseteq \sigma$ if and only if $J \subseteq (\bar{F})R$ and $F \subseteq \sigma$ if and only if $F \in \tilde{\Delta}|_\sigma$. Apply Hochster's formula again to get

$$\beta_{l,\sigma}(R/J) = \widetilde{\mathrm{H}}_{|\sigma|-l-1}(\tilde{\Delta}|_\sigma;\Bbbk) = \widetilde{\mathrm{H}}_{|\sigma|-l-1}(\Delta|_\sigma;\Bbbk) = \beta_{l,\sigma}(R/I).$$

(b): We take the multigraded exact sequence of R-modules:

$$0 \longrightarrow \frac{R}{(I:x)}(-x) \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(I,x)} \longrightarrow 0. \tag{1.3}$$

The corresponding multigraded long exact sequence of Tor is

$$\cdots \longrightarrow \mathrm{Tor}_{l+1}(\Bbbk, \tfrac{R}{(I,x)}) \longrightarrow \mathrm{Tor}_l(\Bbbk, \tfrac{R}{(I:x)}(-x)) \longrightarrow \mathrm{Tor}_l(\Bbbk, \tfrac{R}{I}) \longrightarrow \cdots.$$

Let $W = V \setminus \{x\}$ and $J = (I \cap \Bbbk[W])R$. Since $\beta_{l,\sigma}(R/(I:x)) \neq 0$ and x does not divide any monomial minimal generator of $(I:x)$, we have, by the same argument as in (a), $\sigma \subseteq W$. Let $\tau = \sigma \cup \{x\}$. First observe that

$$\operatorname{Tor}_l\left(\Bbbk, \frac{R}{(I:x)}\right)_\sigma \simeq \operatorname{Tor}_l\left(\Bbbk, \frac{R}{(I:x)}(-x)\right)_\tau.$$

Let us assume that $\beta_{l,\tau}(R/I) = 0$, because, if $\beta_{l,\tau}(R/I) \neq 0$, there is nothing to prove. Then, the above long exact sequence of Tor, restricted to the multidegree τ, implies that $\operatorname{Tor}_{l+1}(\Bbbk, \frac{R}{(I,x)})_\tau \neq 0$. Now , since $(I,x) = (J,x)$, we see further $\operatorname{Tor}_{l+1}(\Bbbk, \frac{R}{(J,x)})_\tau \neq 0$.

Since x is a non-zerodivisor on R/J, we have a multigraded short exact sequence

$$0 \longrightarrow \tfrac{R}{J}(-x) \longrightarrow \tfrac{R}{J} \longrightarrow \tfrac{R}{(J,x)} \longrightarrow 0,$$

which gives the following long exact sequence of Tor:

$$\cdots \longrightarrow \operatorname{Tor}_{l+1}(\Bbbk, \tfrac{R}{J}) \longrightarrow \operatorname{Tor}_{l+1}(\Bbbk, \tfrac{R}{(J,x)}) \longrightarrow \operatorname{Tor}_l(\Bbbk, \tfrac{R}{J}(-x)) \longrightarrow \cdots.$$

Since x does not divide any minimal monomial generator of J, $\beta_{l+1,\tau}(R/J) = 0$. Therefore $\operatorname{Tor}_l(\Bbbk, \frac{R}{J}(-x))_\tau \neq 0$, or, equivalently, $\operatorname{Tor}_l(\Bbbk, \frac{R}{J})_\sigma \neq 0$. By (a) above, $\beta_{l,\sigma}(R/I) \neq 0$. $\square$

Remark 1.3.9. G. Lyubeznik showed that, with notation as above, $\operatorname{depth} R/(I:x) \geq \operatorname{depth} R/I$ [Lyu88a, Lemma 1.1]; Lemma 1.3.8(b) gives another proof.

Chapter 2

Regularity and Depth of Bipartite Edge Ideals

We saw, in Proposition 1.3.4, that questions about numerical invariants of free resolutions of arbitrary monomial ideals can be reduced to the square-free case using polarization. Square-free quadratic monomials can be thought of as edges of a graph; we will use this description to get bounds on the regularity and depth of quadratic monomial ideals. In the last section, we classify all CM bipartite graphs whose edge ideals have quasi-pure resolutions.

2.1 Edge Ideals

Let $I \subseteq R = \Bbbk[V]$ be a square-free quadratic monomial ideal. We can define a graph G on the vertex set V by setting, for all $x, y \in V$, xy to be an (undirected) edge if $xy \in I$. Then G is a simple graph, *i.e.*, it has no loops (edges whose both end points are the same) and no parallel edges (between the same pair of vertices). Conversely, for any simple graph G on the vertex set V, we can define a square-free quadratic monomial ideal, generated by monomials xy whenever there is an edge between x and y, for all $x, y \in V$. We call I the *edge ideal* of G. We will say that G is *unmixed* (respectively, *Cohen-Macaulay*) if R/I is unmixed (respectively, Cohen-Macaulay).

Many invariants of I can be described in terms of invariants of G. The theory of edge ideals is systematically developed in [Vil01, Chapter 6]. A general reference to graph theory is [Bol98].

Let G be a simple graph on V with edge ideal I. A *vertex cover* of G is a set $A \subseteq V$ such that whenever xy is an edge of G, $x \in A$ or $y \in A$. Let A be a vertex cover of G. Then the prime ideal $(x : x \in A) \subseteq R$ contains I. Conversely, if $\mathfrak{p} \subseteq R$ is a monomial prime ideal (hence is generated by some $A \subseteq V$) then A is a vertex cover. Since I is square-free, R/I is reduced; therefore, $\operatorname{Ass} R/I$ is the set of minimal prime ideals containing I. These are monomial ideals, and, hence are in bijective correspondence with the set of *minimal* vertex covers of G. Observe that if G is unmixed, then all its minimal vertex covers have the same size.

If xy is an edge of G, then we say that x and y are *neighbours* of each other. An edge is *incident* on its vertices. We say that an edge xy is *isolated* if there are no other edges incident on x or on y. A vertex x is a *leaf vertex* if there is a unique $y \in V$ such that xy is an edge that is not isolated; in this case, we call y a *stem vertex*, and refer to the edge xy as a *leaf*. The *degree* of a vertex x, denoted $\deg_G x$, is the number of edges incident on x. A *tree* is a connected acyclic graph, and a *forest* is a graph in which each connected component is a tree. A graph G is *bipartite*, if there is a partition $V = V_1 \bigsqcup V_2$ and every edge of G is of the form xy where $x \in V_1$ and $y \in V_2$. A *path* is a tree in which every vertex has degree at most two. A *cycle* is a connected graph in which every vertex has degree exactly two. A graph G is bipartite if and only if it does not contain odd cycles; see, *e.g.*, [Bol98, Chapter 1, p. 9, Theorem 4]. In particular, forests are bipartite.

Let G be a graph. A *matching* in G is a maximal (under inclusion) set $\mathfrak{m}$ of edges such that for all $x \in V$, at most one edge in $\mathfrak{m}$ is incident on x. Edges in a matching form a regular sequence on R. We say that G has *perfect matching*, or, is *perfectly matched*, if there is a matching $\mathfrak{m}$ such that for all $x \in V$, exactly one edge in $\mathfrak{m}$ is incident on x.

Lemma 2.1.1. *Let G be a bipartite graph on the vertex set $V = V_1 \bigsqcup V_2$, with edge ideal I. Then G has a perfect matching if and only if $|V_1| = |V_2| = \operatorname{ht} I$. In particular, unmixed bipartite graphs have perfect matching.*

Proof. If G has a perfect matching, then $|V_1| = |V_2|$. Moreover, by König's theorem (see, *e.g.* [Vil01, Section 6.4]), the maximum size of any matching equals the minimum size of any vertex cover; hence $|V_1| = |V_2| = \operatorname{ht} I$. Conversely, if $|V_1| = |V_2| = \operatorname{ht} I$, then, again by König's theorem, G has a matching of $|V_1| = |V_2|$ edges, *i.e.*, it has a perfect matching.

If G is unmixed, then every minimal vertex cover of G has the same size. Observe that both V_1 and V_2 are minimal vertex covers of G. □

We can restate König's theorem in the language of algebra: the maximum length of a regular sequence in the set of monomial minimal generators of the edge ideal equals the height of the ideal.

A graph G is said to be the *suspension* of a subgraph G', if G is obtained by attaching exactly one leaf vertex to every vertex of G'. If G is the suspension of a subgraph G', then G is Cohen-Macaulay. We see this as follows. Let $V' \subseteq V$ be the set of vertices of G'. Denote the edge ideal of G' in $\Bbbk[V']$ by I'. Then I is the polarization of $I' + (x^2 : x \in V')$ in the ring $R = \Bbbk[V]$. Since $\Bbbk[V']/(I' + (x^2 : x \in V'))$ is Artinian, we see that R/I is Cohen-Macaulay. Villarreal showed that a tree G is Cohen-Macaulay if and only if G is the suspension of a subgraph G'; see, *e.g.*, [Vil01, Theorem 6.5.1].

2.2 Bipartite Edge Ideals

In Lemma 2.1.1, we saw that unmixed bipartite graphs have perfect matching. In this section, we will describe the regularity and depth of edge ideals of unmixed bipartite

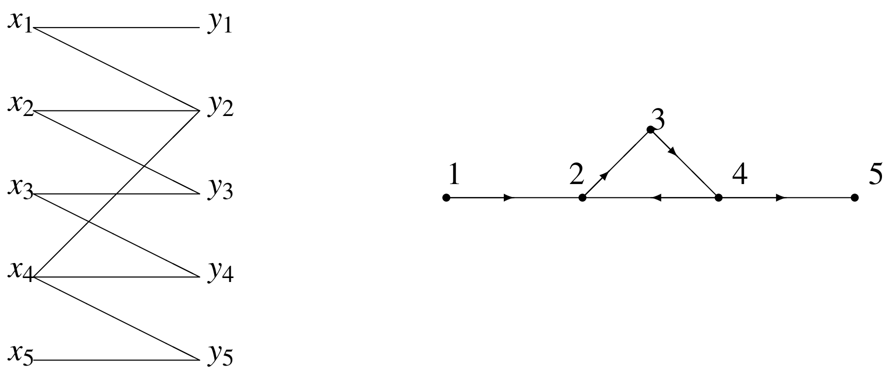


Figure 2.1: A perfectly matched bipartite graph and the associated directed graph

graphs. In Section 4.6, we will use the set-up below to prove Conjecture (HHSu) (see Section 4.1, p. 69) for bipartite edge ideals.

Discussion 2.2.1. Let G be a bipartite graph on $V = V_1 \bigsqcup V_2$ with perfect matching. Let $V_1 = \{x_1, \cdots, x_c\}$ and $V_2 = \{y_1, \cdots, y_c\}$. After relabelling the vertices, we will assume that $x_i y_i$ is an edge for all $i \in [c]$. We associate G with a directed graph $\mathfrak{d}_G$ on $[c]$ defined as follows: for $i, j \in [c]$, ij is an edge of $\mathfrak{d}_G$ if and only if $x_i y_j$ is an edge of G. (Here, by ij, we mean the directed edge from i to j.) We will write $j \succ i$ if there is a directed path from i to j in $\mathfrak{d}$. By $j \succcurlyeq i$ (and, equivalently, $i \preccurlyeq j$) we mean that $j \succ i$ or $j = i$. An example of a perfectly matched bipartite graph and the directed graph associated to it is shown in Figure 2.1. For $A \subseteq [c]$, we say that $j \succcurlyeq A$ if there exists $i \in A$ such that $j \succcurlyeq i$. Let $\mathfrak{d}$ be any directed graph on $[c]$, and denote the underlying undirected graph of $\mathfrak{d}$ by $|\mathfrak{d}|$. A vertex i of $\mathfrak{d}$ is called a *source* (respectively, *sink*) vertex if it has no edge directed towards (respectively, away from) it. We say that a set $A \subseteq [c]$ is an *antichain* if for all $i, j \in A$, there is no directed path from i to j in $\mathfrak{d}$, and, by $\mathscr{A}_{\mathfrak{d}}$, denote the set of antichains in $\mathfrak{d}$. We consider $\emptyset$ as an antichain. A *coclique* of $|\mathfrak{d}|$ is a set $A \subseteq [c]$ such that for all $i \neq j \in A$, i and j are not neighbours in $|\mathfrak{d}|$. Antichains in $\mathfrak{d}$ are cocliques in $|\mathfrak{d}|$, but the converse is not, in general, true. We say that $\mathfrak{d}$ is *acyclic* if there are no directed cycles, and *transitively closed* if, for all $i, j, k \in [c]$, whenever ij and jk are (directed) edges in $\mathfrak{d}$, ik is an edge. Observe that $\mathfrak{d}$ is a poset under the order $\succcurlyeq$ if (and only if) it is acyclic and transitively closed. In this case, for all $A \subseteq [c]$, A is an antichain in $\mathfrak{d}$ if and only if A is a coclique in $|\mathfrak{d}|$. Let $\kappa(G)$ denote the largest size of any coclique in $|\mathfrak{d}_G|$. If $|\mathfrak{d}_G|$ is a poset, we say that, for $i, j \in [c]$, j *covers* i if $j \succ i$ and there does not exist j' such that $j \succnsim j' \succnsim i$. (A general reference for results on posets is [Sta97, Chapter 3].) $\square$

The significance of $\kappa(G)$ is that it gives a lower bound for $\operatorname{reg} R/I$. Following Zheng [Zhe04], we say that two edges vw and $v'w'$ of a graph G are *disconnected* if they are no more edges between the four vertices v, v', w, w'. A set $\mathbf{a}$ of edges is pairwise disconnected if and only if $(I \cap \Bbbk[V_{\mathbf{a}}])R$ is generated by the regular sequence of edges in $\mathbf{a}$, where by $V_{\mathbf{a}}$, we mean the set of vertices on which the edges in $\mathbf{a}$ are incident. The latter condition holds if and only if the subgraph of G induced on $V_{\mathbf{a}}$, denoted as $G|_{V_{\mathbf{a}}}$, is a collection of $|\mathbf{a}|$ isolated edges. In particular, the edges in any pairwise disconnected set form a regular sequence in R. Set $r(I) := \max\{|\mathbf{a}| :$ $\mathbf{a}$ is a set of pairwise disconnected edges in $G\}$. We list some results relating $\operatorname{reg} R/I$ and $r(I)$.

Proposition 2.2.2. *Let G be a graph on V with edge ideal I and $\overline{m}_l = \overline{m}_l(R/I)$ for $1 \leq l \leq \operatorname{pd} R/I$.*

a. [Zhe04, Theorem 2.18] If G is a forest, then $\operatorname{reg} R/I = r(I)$.

b. (Essentially from [Kat06, Lemma 2.2]) For $1 \leq l \leq r(I)$, $\overline{m}_l = 2l$ and for $r(I) \leq l \leq c$, $\overline{m}_l \geq l + r(I)$. In particular, $\operatorname{reg} R/I \geq r(I)$.

Lemma 2.2.3. *Let G be bipartite graph with perfect matching. Then, with notation as in Discussion 2.2.1, $r(I) \geq \kappa(G) \geq \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\}$.*

Proof. If $A \subseteq [c]$ is a coclique of $|\mathfrak{d}_G|$, we easily see that the edges $\{x_iy_i : i \in A\}$ are pairwise disconnected in G. The assertion now follows from the observation, that we made in Discussion 2.2.1, that any antichain in $\mathfrak{d}_G$ is a coclique of $|\mathfrak{d}_G|$. □

We will see in Theorem 2.2.15 that when G is a an unmixed bipartite graph, $\operatorname{reg} R/I = r(I)$. In the next section, we will discuss some examples where $\operatorname{reg} R/I > r(I)$. First, we relate some properties of bipartite graphs with their associated directed graphs.

Lemma 2.2.4. *Let G be bipartite graph with perfect matching, and adopt the notation of Discussion 2.2.1. Let $j \succcurlyeq i$. Then for all $\mathfrak{p} \in \operatorname{Unm} R/I$, if $y_i \in \mathfrak{p}$, then $y_j \in \mathfrak{p}$.*

Proof. Applying induction on the length of a directed path from i to j, we may assume, without loss of generality, that ij is a directed edge of $\mathfrak{d}_G$. Let $\mathfrak{p} \in \operatorname{Unm} R/I$ and $k \in [c]$. Since $x_ky_k \in I$, $x_k \in \mathfrak{p}$ or $y_k \in \mathfrak{p}$. Since $\operatorname{ht} \mathfrak{p} = c$, in fact, $x_k \in \mathfrak{p}$ if and only if $y_k \notin \mathfrak{p}$. Now since $y_i \in \mathfrak{p}$, $x_i \notin \mathfrak{p}$, so $(I : x_i) \subseteq \mathfrak{p}$. Note that since x_iy_j is an edge of G, $y_j \in (I : x_i)$. □

Villarreal gave a characterization of unmixed bipartite edge ideals; the following is its restatement in the above set-up.

Proposition 2.2.5 (Villarreal [Vil07, Theorem 1.1])**.** *Let G be a bipartite graph. Then G is unmixed if and only if $\mathfrak{d}_G$ is transitively closed.* □

Discussion 2.2.6. Let $\mathfrak{d}$ be a directed graph. We say that a pair i, j of vertices $\mathfrak{d}$ are *strongly connected* if there are directed paths from i to j and from j to i; see [Wes96, Definition 1.4.12]. A *strong component* of $\mathfrak{d}$ is an induced subgraph maximal under the property that every pair of vertices in it is strongly connected. for more on strong connectivity. Strong components of $\mathfrak{d}$ form a partition of its vertex set. Now let G be a an unmixed bipartite graph. Since $\mathfrak{d}_G$ is transitively closed, we see that for all $i, j \in [c]$, i and j belong to the same strong component of $\mathfrak{d}_G$ if and only if both ij and ji are directed edges. Let $\mathscr{Z}_1, \ldots, \mathscr{Z}_t$ be the vertex sets of the strong components of $\mathfrak{d}_G$. Let

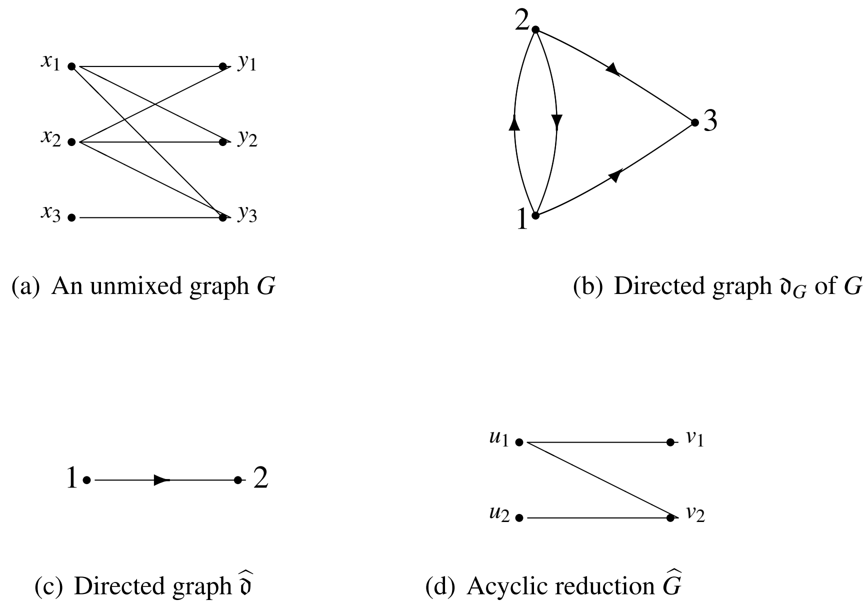
$\zeta_i = |\mathscr{Z}_i|, 1 \leq i \leq t$. Define a directed graph $\widehat{\mathfrak{d}}$ on $[t]$ by setting, for $a, b \in [t]$, ab to be a directed edge (from a to b) if there exists a directed edge in $\mathfrak{d}_G$ from any (equivalently, all, since $\mathfrak{d}_G$ is transitively closed) of the vertices in $\mathscr{Z}_a$ to any (equivalently, all, since $\mathfrak{d}_G$ is transitively closed) of the vertices in $\mathscr{Z}_b$. We observe that $\widehat{\mathfrak{d}}$ has no directed cycles; this follows at once from the definition of the $\mathscr{Z}_a$ and the fact that $\mathfrak{d}_G$ is transitively closed. Further, $\widehat{\mathfrak{d}}$ is transitively closed. Therefore, it is poset under the order induced from $\mathfrak{d}_G$. We will use the same notation for the induced order, *i.e.*, say that $b \succ a$ if there is a directed edge from a to b. Define the *acyclic reduction* of G to be the bipartite graph $\widehat{G}$ on new vertices $\{u_1, \ldots, u_t\} \bigsqcup \{v_1, \ldots, v_t\}$, with edges $u_a v_a$, for all $1 \leq a \leq t$ and $u_a v_b$, for all directed edges ab of $\widehat{\mathfrak{d}}$. Let $S = \Bbbk[u_1, \ldots, u_t, v_1, \ldots, v_t]$, with standard grading. Let $\widehat{I} \subseteq S$ be the edge ideal of $\widehat{G}$. For a multidegree $\sigma = \prod_i u_i^{s_i} \prod v_i^{t_i}$, set $\sigma^\zeta = \prod_i u_i^{s_i \zeta_i} \prod v_i^{t_i \zeta_i}$.

Example 2.2.7. Let G be the bipartite graph in Figure 2.2(a). on $\{x_1, x_2, x_3\} \bigsqcup \{y_1, y_2, y_3\}$. The associated directed graph $\mathfrak{d}_G$ is shown in Figure 2.2(b); its vertex set has the partition $\mathscr{Z}_1 = \{1, 2\}$, $\mathscr{Z}_2 = \{3\}$ into maximal directed cycles. This gives $\widehat{\mathfrak{d}}$, given in Figure 2.2(c). The directed edge 12 of $\widehat{\mathfrak{d}}$ corresponds to the edges 13 and 23 of $\mathfrak{d}_G$. The acyclic reduction $\widehat{G}$ of G is given in Figure 2.2(d).

Lemma 2.2.8. *Let G be an unmixed bipartite graph with edge ideal I. For an antichain $A \neq \varnothing$ of $\widehat{\mathfrak{d}}$, let $\Omega_A = \{j \in \mathscr{Z}_b : b \succcurlyeq A\}$. Let $\Omega_\varnothing = \varnothing$. Then* $\operatorname{Ass} R/I = \{(x_i : i \notin \Omega_A) + (y_i : i \in \Omega_A) : A \in \mathscr{A}_{\widehat{\mathfrak{d}}}\}$.

Proof. Let $\mathfrak{p} \in \operatorname{Ass} R/I$. Let $\mathrm{U} := \{b : y_j \in \mathfrak{p} \text{ for some } j \in \mathscr{Z}_b\}$. It follows from Lemma 2.2.4 that $y_j \in \mathfrak{p}$ for all $j \in \bigcup_{b \in \mathrm{U}} \mathscr{Z}_b$ and that if $b' \succ b$ for some $b \in \mathrm{U}$, then $b' \in \mathrm{U}$. Now, the minimal elements of U form an antichain A under $\succ$. Hence $\{j : y_j \in \mathfrak{p}\} = \Omega_A$, showing $\operatorname{Ass} R/I \subseteq \{(x_i : i \notin \Omega_A) + (y_i : i \in \Omega_A) : A \in \mathscr{A}_{\widehat{\mathfrak{d}}}\}$.

Conversely, let $A \in \mathscr{A}_{\widehat{\mathfrak{d}}}$ and $\mathfrak{p} := (x_i : i \notin \Omega_A) + (y_i : i \in \Omega_A)$. Since $\operatorname{ht} \mathfrak{p} = c = \operatorname{ht} I$, it suffices to show that $I \subseteq \mathfrak{p}$ in order to show that $\mathfrak{p} \in \operatorname{Ass} R/I$. Clearly, for all $1 \leq i \leq c$,



(a) An unmixed graph G

(b) Directed graph $\mathfrak{d}_G$ of G

(c) Directed graph $\widehat{\mathfrak{d}}$

(d) Acyclic reduction $\widehat{G}$

Figure 2.2: Acyclic Reduction, Example 2.2.7

$x_iy_i \in \mathfrak{p}$. Take $i \neq j$ such that $x_iy_j \in I$. If $i \notin \Omega_A$, then there is nothing to show. If $i \in \Omega_A$, then there exist a, b, b' such that $a \in A$, $b \succ a$, $i \in \mathscr{Z}_b$ and $j \in \mathscr{Z}_{b'}$. Since ij is a directed edge of $\mathfrak{d}_G$, $b' \succ b$ in $\widehat{\mathfrak{d}}$. Hence $b' \succ a$, and $j \in \Omega_A$, giving $y_j \in \mathfrak{p}$. This shows that $I \subseteq \mathfrak{p}$. $\square$

Proposition 2.2.9. *Let G be an unmixed bipartite graph, with edge ideal I and acyclic reduction $\widehat{G}$. Let $\widehat{I} \subseteq S$ be the edge ideal of $\widehat{G}$. Then* $\operatorname{reg} R/I = \operatorname{pd}\left(\widehat{I}\right)^\star$ *and* $\operatorname{pd} R/I = \max\{|\sigma^\zeta| - l : \beta_{l,\sigma}\left(\left(\widehat{I}\right)^\star\right) \neq 0\}$.

Proof. By Proposition 1.3.6, $\operatorname{reg} R/I = \operatorname{pd} I^\star$ and $\operatorname{pd} R/I = \operatorname{reg} I^\star$. Hence it suffices to show that $\operatorname{pd} I^\star = \operatorname{pd}\left(\widehat{I}\right)^\star$ and $\operatorname{reg} I^\star = \max\{|\sigma^\zeta| - l : \beta_{l,\sigma}\left(\left(\widehat{I}\right)^\star\right) \neq 0\}$. From Lemma 2.2.8, with the notation used there, it follows that

$$I^\star = \left(\prod_{i \notin \Omega_A} x_i \cdot \prod_{i \in \Omega_A} y_i : A \in \mathscr{A}_{\widehat{\mathfrak{d}}}\right) = \left(\prod_{\substack{b \not\succeq A \\ i \in \mathscr{Z}_b}} x_i \cdot \prod_{\substack{b \succeq A \\ i \in \mathscr{Z}_b}} y_i : \varnothing \neq A \in \mathscr{A}_{\widehat{\mathfrak{d}}}\right) + \left(\prod_{i=1}^{c} x_i\right).$$

For each $a \in [t]$, fix $i_a \in \mathscr{Z}_a$. Now, as the $\mathscr{Z}_a$ form a partition of $[c]$, we see that $I^\star$ is a polarization of the ideal

$$J = \left(\prod_{b \not\geq A} x_{i_b}^{\zeta_b} \cdot \prod_{b \geq A} y_{i_b}^{\zeta_b} : \varnothing \neq A \in \mathscr{A}_{\widehat{\mathfrak{d}}} \right) + \left(\prod_{b=1}^{t} x_{i_b}^{\zeta_b} \right).$$

Hence it suffices to show that $\operatorname{pd} I^\star = \operatorname{pd} \left(\widehat{I}\right)^\star$ and $\operatorname{reg} I^\star = \max\{|\sigma^\zeta| - l : \beta_{l,\sigma}\left(\left(\widehat{I}\right)^\star\right) \neq 0\}$. The following lemma now concludes the proof. $\square$

Lemma 2.2.10. *Let $B_1 = \Bbbk[x_1, \ldots, x_n]$ and $B_2 = \Bbbk[y_1, \ldots, y_n]$. Let $\xi_1, \ldots, \xi_n$ be positive integers. Set $\deg x_i = 1$ and $\deg y_i = \xi_n$ for all $1 \leq i \leq n$. Define a ring homomorphism $\phi : B_2 \longrightarrow B_1$ by sending $y_i \mapsto x_i^{\xi_i}$. Then for any acyclic complex $\mathbb{G}_\bullet$ of finitely generated graded B_2-modules (with degree-preserving maps), $\mathbb{G}_\bullet \otimes_{B_2} B_1$ is an acyclic complex of finitely generated graded B_1-modules (with degree-preserving maps).*

Proof. Acyclicity of $\mathbb{G}_\bullet \otimes_{B_2} B_1$ follows from the fact that B_1 is a free and hence flat B_2-algebra. The maps in $\mathbb{G}_\bullet \otimes_{B_2} B_1$ are degree-preserving since ϕ preserves degrees. $\square$

Remark 2.2.11. Let G be an unmixed graph with acyclic reduction $\widehat{G}$. If $I \subseteq R$ and $\widehat{I} \subseteq S$ are the respective edge ideals, then it follows from Proposition 2.2.9, that $\operatorname{reg} R/I = \operatorname{pd}\left(\widehat{I}\right)^\star = \operatorname{reg} S/\widehat{I}$.

Herzog and T. Hibi gave the following characterization of Cohen-Macaulay bipartite graphs.

Theorem 2.2.12. *[HH05, Lemma 3.3 and Theorem 3.4] Let G be a bipartite graph on $V_1 \bigsqcup V_2$, with edge ideal I. Then G is Cohen-Macaulay if and only if $|V_1| = |V_2| = c = \operatorname{ht} I$ and we can write $V_1 = \{x_1, \cdots, x_c\}$ and $V_2 = \{y_1, \cdots, y_c\}$ such that*

a. For all $1 \leq i \leq n$, $x_i y_i$ is an edge of G.

b. For all $1 \leq i, j \leq n$, if $x_i y_j$ is an edge of G, then $j \geq i$.

c. *For all* $1 \le i, j, k \le n$, *if* $x_i y_j$ *and* $x_j y_k$ *are edges of* G, *then* $x_i y_k$ *is an edge of* G.

Below we first paraphrase this result in terms of the associated directed graph and give an alternate proof.

Theorem 2.2.13. *Let* G *be a bipartite graph on the vertex set* $V = V_1 \bigsqcup V_2$. *Then* G *is Cohen-Macaulay if and only if* G *is perfectly matched and the associated directed graph* $\mathfrak{d}_G$ *is acyclic and transitively closed, i.e., it is a poset.*

Proof. Let I be the edge ideal of G. First assume that G is Cohen-Macaulay; then G is unmixed, so, by Proposition 2.2.5, G is perfectly matched and $\mathfrak{d}_G$ is transitively closed. Let $[c] = \mathscr{Z}_1 \bigsqcup \ldots \bigsqcup \mathscr{Z}_t$ be a partition of $[c]$ into vertex sets of maximal directed cycles, as in Discussion 2.2.6. Let $\zeta_a = |\mathscr{Z}_a|, 1 \le a \le t$. Let S and $\widehat{I}$ be as in Discussion 2.2.6. We need to show that $\zeta_a = 1$ for all $1 \le a \le t$ (which implies that $t = c$).

Since $\operatorname{pd} R/I = c = \operatorname{ht} I$, from Theorem 2.2.9 we see that $\operatorname{reg}(\widehat{I})^\star = c$. Notice that $(\widehat{I})^\star$ is minimally generated by monomials of degree c (in the grading of S). For any $a \in [t]$,

$$\prod_{b \not\succcurlyeq a} u_b \cdot \prod_{b \succcurlyeq a} v_b \text{ and } \prod_{b \preccurlyeq a} u_b \cdot \prod_{b \succ a} v_b$$

are minimal monomial generators of $(\widehat{I})^\star$. It is easy to see that

$$u_a \left(\prod_{b \not\succcurlyeq a} u_b \cdot \prod_{b \succcurlyeq a} v_b \right) - v_a \left(\prod_{b \preccurlyeq a} u_b \cdot \prod_{b \succ a} v_b \right) = 0$$

is a minimal syzygy of the generators of $(\widehat{I})^\star$. The degree of this syzygy is $c + \deg u_a = c + \deg v_a = c + \zeta_a$. Since $c + \zeta_a - 1 \le \operatorname{reg}(\widehat{I})^\star = c$, we see that $\zeta_a \le 1$, which gives $\zeta_a = 1$.

Conversely, assume that G is perfectly matched and that $\mathfrak{d}_G$ is acyclic and transitively closed. We prove the assertion by induction on $c = \operatorname{ht} I = |V_1| = |V_2|$. If $c = 1$, then

$I=(x_1y_1)$, and the assertion easily follows. Now let $c\geq 2$. Observe that $\dim R/I=c$. We need to show that $\operatorname{depth} R/I=c$. By Proposition 2.2.5, G is unmixed. Notice that $\mathfrak{d}_G$ is a poset, under the order $\succcurlyeq$, as mentioned in Discussion 2.2.1.

Let $i\in[c]$ be a minimal element of $\mathfrak{d}_G$. Observe that it suffices to show that $\operatorname{depth} R/(I:x_i)=c=\operatorname{depth} R/(I,x_i)$; then, it follows from the exact sequence

$$0 \longrightarrow \frac{R}{(I:x_i)}(-x_i) \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(I,x_i)} \longrightarrow 0$$

that $\operatorname{depth} R/I=c$.

First, since i is a minimal element of $\mathfrak{d}_G$, y_i is a leaf vertex G. Write $W=\{x_j : j\in[c], j\neq i\}\cup\{y_j : j\in[c], j\neq i\}$. It is easy to see that $(I,x_i)=(I\cap\Bbbk[W])R+(x_i)$, and that x_i is a non-zero-divisor on $R/(I\cap\Bbbk[W])R$. Moreover, (I,x_i) is the edge ideal of $G|_W$, which is perfectly matched. The associated directed graph of $G|_W$ (on the vertex set $[c]\setminus\{i\}$) is the sub-poset $\mathfrak{d}':=\{j\in\mathfrak{d}_G : j\neq i\}$ of $\mathfrak{d}_G$. Since $\mathfrak{d}'$ is transitively closed and has no directed cycles, we see, by induction, that $G|_W$ is Cohen-Macaulay with $\operatorname{ht}(I\cap\Bbbk[W])=c-1$, giving that $R/(I,x_i)$ is Cohen-Macaulay with $\operatorname{depth} R/(I,x_i)=c$.

Secondly, we can write $(I:x_i)=(y_j : j\in[c], j\succcurlyeq i)+\tilde{J}$, where we set $\tilde{\mathfrak{d}}=\{j\in[c] : j\not\succcurlyeq i\}$ and $\tilde{J}$ is the edge ideal (in R) of the induced subgraph $\tilde{G}$ of G on the subset of vertices $\{x_j,y_j : j\in\tilde{\mathfrak{d}}\}$. For, that $(I:x_i)\supseteq(y_j : j\in[c], j\succcurlyeq i)+\tilde{J}$ is clear. To show the other inclusion, if $x_jy_k\in I$ for some $j\succcurlyeq i$, then $x_iy_k\in I$, and, therefore, $y_k\in(I:x_i)$, showing that x_jy_k is not a minimal generator of $(I:x_i)$. If i is the unique minimal element of $\mathfrak{d}_G$, then we take $\tilde{J}=0$. Moreover, $y_j, j\succcurlyeq i$ is a regular sequence on $R/\tilde{J}$. Indeed, $\tilde{\mathfrak{d}}$ is the associated directed graph of $\tilde{G}$, so, by induction, $R/\tilde{J}$ is Cohen-Macaulay with $\dim R/\tilde{J}=c+|\{j : j\succcurlyeq i\}|$. Hence $R/(I:x_i)$ is Cohen-Macaulay and $\operatorname{depth} R/(I:x_i)=c$. $\square$

Lemma 2.2.14. *Let G be an unmixed bipartite graph with acyclic reduction $\widehat{G}$. Then* $\max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_G}\} = \max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_{\widehat{G}}}\}$.

Proof. Let $A = \{i_1, \ldots, i_r\} \subseteq [c]$ be an antichain in $\mathfrak{D}_G$. Choose $a_1, \ldots, a_r \in [t]$ such that $i_j \in \mathscr{Z}_{a_j}$. Since $\mathfrak{D}_G$ is transitively closed, it follows that $\{a_1, \ldots, a_r\}$ is an antichain in $\mathfrak{D}_{\widehat{G}}$. Conversely, if $\{a_1, \ldots, a_r\}$ is an antichain in $\mathfrak{D}_{\widehat{G}}$, then for any choice of $i_j \in \mathscr{Z}_{a_j}$, $\{i_1, \ldots, i_r\}$ is an antichain in $\mathfrak{D}_G$. $\square$

Theorem 2.2.15. *Let G be an unmixed bipartite graph with edge ideal I. Then* $\operatorname{reg} R/I = \max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_G}\}$. *In particular,* $\operatorname{reg} R/I = r(I)$.

Proof. The latter statement follows from the first statement along with Lemma 2.2.3. In order to prove the first statement, let $\widehat{G}$ be the acyclic reduction of G on the vertex set $\{u_1, \ldots, u_t\} \bigsqcup \{v_1, \ldots, v_t\}$. Recall that $\widehat{G}$ is a Cohen-Macaulay bipartite graph. As in Discussion 2.2.6, let $S = \Bbbk[u_1, \ldots, u_t, v_1, \ldots, v_t]$. Let $\widehat{I} \subseteq S$ to be the edge ideal of $\widehat{G}$. Remark 2.2.11 and Lemma 2.2.14 give that it suffices to prove the theorem for Cohen-Macaulay bipartite graphs.

After relabelling, we take G to be Cohen-Macaulay; then $\mathfrak{D}_G$ is a poset. We proceed by induction on c. If $c = 1$, then G is x_1y_1, for which the theorem is true. Hence let $c > 1$. Let $i \in [c]$ be a minimal element of $\mathfrak{D}_G$. First, the deletion $\mathfrak{D}_1$ of i in $\mathfrak{D}_G$ is the associated directed graph of the deletion G_1 of the vertices x_i and y_i (which is a leaf vertex) in G. Since $\mathfrak{D}_1$ is a poset, G_1 is Cohen-Macaulay, on $2(c-1)$ vertices. Let I_1 be the edge ideal of G_1. Any antichain of $\mathfrak{D}_1$ is an antichain of $\mathfrak{D}_G$; so, by induction, $\operatorname{reg} R/I_1 \leq \max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_G}\}$. Secondly, let $\mathfrak{D}_2$ be the poset $\{j \in [c] : j \not\succeq i\}$. Let G_2 be the Cohen-Macaulay bipartite graph whose associated directed graph is $\mathfrak{D}_2$. Let I_2 be the edge ideal of G_2. For any antichain A of $\mathfrak{D}_2$, we see that $A \cup \{i\}$ is an antichain of $\mathfrak{D}_G$. Hence, by induction, $\operatorname{reg} R/I_2 \leq \max\{|A| : A \in \mathscr{A}_{\mathfrak{D}_G}\} - 1$.

Now, $(I,x_i) = (I_1,x_i)$ and $(I : x_i) = I_2 + (y_j : j \succcurlyeq i)$. Since x_i is a non-zero-divisor of R/I_1 and $\{y_j, j \succcurlyeq i\}$ is a regular sequence on R/I_2, we see that $\operatorname{reg} \frac{R}{(I,x_1)} \leq \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\}$ and that $\operatorname{reg} \frac{R}{(I:x_1)} \leq \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\} - 1$ It now follows from the exact sequence

$$0 \longrightarrow \frac{R}{(I:x_i)}(-x_i) \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(I,x_i)} \longrightarrow 0$$

that $\operatorname{reg} R/I = \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\}$. □

Remark 2.2.16. Let G be a Cohen Macaulay bipartite graph with edge ideal I, with $\operatorname{ht} I = c$. Then $\operatorname{reg} R/I \leq c$. If $\operatorname{reg} R/I = c$, then R/I is a complete intersection, or, equivalently, G consists of c isolated edges. We see this as below: Let $\mathfrak{d}_G$ be the associated directed graph on $[c]$. Since $\operatorname{reg} R/I$ is the maximum size of an antichain in $\mathfrak{d}_G$, $\operatorname{reg} R/I \leq c$. If $\operatorname{reg} R/I = c$, we see that $\mathfrak{d}_G$ has an antichain of c elements, which implies that for all $i \neq j \in [c]$, $i \not\succcurlyeq j$ or $j \not\succcurlyeq i$, *i.e.*, $x_i y_j$ is not an edge of G. □

We would now like to give a description of $\operatorname{depth} R/I$ for an unmixed bipartite edge ideal I in terms of the associated directed graph. First, we determine the multidegrees with non-zero Betti numbers for its Alexander dual.

Let G be a Cohen-Macaulay bipartite graph. For antichains $B \subseteq A$ of $\mathfrak{d}_G$, $A \neq \varnothing$, set $\sigma_{A,B} := \prod_{i \not\succcurlyeq A} x_i \prod_{i \succcurlyeq A} y_i \prod_{i \in B} x_i$. Set $\sigma_{\varnothing,\varnothing} = \prod_{i=1}^{c} x_i$.

Theorem 2.2.17. *Let G be a Cohen-Macaulay bipartite graph with edge ideal I. For all $l \geq 0$, and multidegrees σ, if $\beta_{l,\sigma}(I^\star) \neq 0$, then $\beta_{l,\sigma}(I^\star) = 1$ and $\sigma = \sigma_{A,B}$ for some antichains $B \subseteq A$ of $\mathfrak{d}_G$ with $|B| = l$.*

Proof. We assume that $l \geq 1$ since the conclusion for the case $l = 0$ follows from Lemma 2.2.8. (Since G is Cohen-Macaulay, $\widehat{\mathfrak{d}} = \mathfrak{d}_G$.) Let l, σ be such that $\beta_{l,\sigma}(I^\star) \neq 0$. Recall that $I^\star$ has a linear resolution, so $|\sigma| = c + l$ and $|\bar{\sigma}| = c - l$.

Let A be the set of the minimal elements of the set $\{i \in [c] : y_j | \sigma \text{ for all } j \succcurlyeq i\}$. It is a non-empty antichain, because σ is divisible by a minimal generator of $I^\star$ different from $\prod_{i=1}^{c} x_i$; any such generator is $\sigma_{A',\varnothing}$ for some antichain $A' \neq \varnothing$. We now claim that for all $i \not\succcurlyeq A$, $x_i | \sigma$. For, if not, then we see from Lemma 1.3.8(a) that $\beta_{l,\sigma}(I^\star \cap \Bbbk[V \setminus \{x_i\}]) = \beta_{l,\sigma}((I : x_i)^\star) \neq 0$. However, $y_j \in (I : x_i)$, for all $j \succcurlyeq i$, giving $y_j | \sigma$ for all $j \succcurlyeq i$. Hence $i \succcurlyeq A$, contradicting the hypothesis that $i \not\succcurlyeq A$.

Define $B_1 = \{i : i \succcurlyeq A, x_i | \sigma\}$ and $B_2 = \{i : i \not\succcurlyeq A, y_i | \sigma\}$. We observe that for all $i \in B_2$, there exists j such that $j \succcurlyeq i$ and $y_j \nmid \sigma$; for, otherwise, it will follow from the definition of A that $i \succcurlyeq A$, which is a contradiction. Also note that $|B_1| + |B_2| = l$. We need to show that $B_1 \subseteq A$ and that $B_2 = \varnothing$. Further, define $B_1' = \{j \in B_1 : i \in B_1 \text{ for all } i \text{ such that } j \succcurlyeq i \succcurlyeq A\}$.

We now observe that $x_i \notin (I : \bar{\sigma})$ if and only if for all $j \succcurlyeq i$, $y_j \nmid \bar{\sigma}$ if and only if for all $j \succcurlyeq i$, $y_j | \sigma$ if and only if for all $j \succcurlyeq i$, $j \succcurlyeq A$ or $j \in B_2$. This condition holds if $i \succcurlyeq A$. If $i \not\succcurlyeq A$, then, since $y_i | \sigma$, $i \in B_2$; however, in this case, we noted earlier that there exists $j \succcurlyeq i$ such that $y_j \nmid \sigma$. Thus $x_i \notin (I : \bar{\sigma})$ if and only if $i \succcurlyeq A$. Similarly, $y_j \notin (I : \bar{\sigma})$ if and only if for all $i \preccurlyeq j$, $x_i \nmid \bar{\sigma}$ if and only if for all $i \preccurlyeq j$, $x_i \mid \sigma$ if and only if for all $i \preccurlyeq j$, $i \not\succcurlyeq A$ or $i \in B_1$ if and only if $j \not\succcurlyeq A$ or $j \in B_1'$.

Now, $x_i y_j$ is a minimal monomial generator of $(I : \bar{\sigma})$ if and only if $x_i \notin (I : \bar{\sigma})$ and $y_j \notin (I : \bar{\sigma})$ if and only if $(i \succcurlyeq A)$ and $(j \not\succcurlyeq A$ or $j \in B_1')$, which, because $j \succcurlyeq i$, is equivalent to the condition that $i, j \in B_1'$. Let $J := (x_i y_j : i, j \in B_1')$. Let $\tau = \bigcup_{i \in B_1'} \{x_i, y_i\}$. We can then write $(I : \bar{\sigma}) = JR + (\sigma \setminus \tau)R$. Since minimal generators of $(\sigma \setminus \tau)$ form a regular sequence of linear forms on R/J, we see that

$$\frac{R}{(I : \bar{\sigma})} \simeq \frac{\Bbbk[\tau]}{J\Bbbk[\tau]} \otimes_\Bbbk \frac{\Bbbk[\sigma \setminus \tau]}{(\sigma \setminus \tau)\Bbbk[\sigma \setminus \tau])} \otimes_\Bbbk \Bbbk[\bar{\sigma}]. \tag{2.1}$$

In particular, R/J is Cohen-Macaulay and $\operatorname{reg} R/J = \operatorname{reg} \frac{R}{(I:\bar{\sigma})}$. We know from (1.2) that $\operatorname{reg} \frac{R}{(I:\bar{\sigma})} \geq l$. Since J is the edge ideal, in R, of the restriction $G|_\tau$ of G to the vertex set τ, we see that $\operatorname{reg} R/J$ is the maximum size of an antichain of the restriction of $\mathfrak{d}_G$ to B'_1. Hence $|B'_1| \geq l$. However, $B'_1 \subseteq B_1$ and $|B_1| + |B_2| = l$, so $B'_1 = B_1$ and $B_2 = \varnothing$. Furthermore, so as to have an antichain of size l, any two elements of $B'_1 = B_1$ must be incomparable to each other, so $B_1 \subseteq A$. Set $B = B_1$. This proves that $\sigma = \sigma_{A,B}$, for some antichains $B \subseteq A$ with $|B| = l$.

Finally, to show that $\beta_{l,\sigma}(I^\star) = 1$, we observe that $G|_\tau$ is a collection of l isolated edges, so R/J is a complete intersection and $\beta_{l,\tau}(R/J) = 1$. (Note that $|\tau| = 2l$ and that $\tau \subseteq \sigma$.) Recall that $|\sigma| = c + l$. Now, from (1.2) and (2.1), it follows that

$$\begin{aligned}
\beta_{l,\sigma}(I^\star) &= \beta_{c,\sigma}\left(\frac{R}{(I:\bar{\sigma})}\right) \\
&= \sum_{0 \leq i \leq c} \sum_{\tau' \subseteq \sigma} \beta_{i,\tau'}\left(\frac{\mathbb{k}[\tau]}{J\mathbb{k}[\tau]}\right) \beta_{c-i,\sigma\setminus\tau'}\left(\frac{\mathbb{k}[\sigma\setminus\tau]}{(\sigma\setminus\tau)\mathbb{k}[\sigma\setminus\tau]}\right) \\
&= \sum_{0 \leq i \leq c} \beta_{i,\tau}\left(\frac{\mathbb{k}[\tau]}{J\mathbb{k}[\tau]}\right) \beta_{c-i,\sigma\setminus\tau}\left(\frac{\mathbb{k}[\sigma\setminus\tau]}{(\sigma\setminus\tau)\mathbb{k}[\sigma\setminus\tau]}\right) \\
&\qquad \big(\text{summands above with } \tau' \not\subseteq \tau, \sigma\setminus\tau' \not\subseteq \sigma\setminus\tau \text{ are zero}\big) \\
&= \beta_{l,\tau}\left(\frac{\mathbb{k}[\tau]}{J\mathbb{k}[\tau]}\right) \qquad (\text{from the Koszul resolution of } \mathbb{k} \text{ over } \mathbb{k}[\sigma\setminus\tau]) \\
&= \beta_{l,\tau}(R/J) \qquad (\text{since } \mathbb{k}[\tau] \to R \text{ is faithfully flat}) \\
&= 1.
\end{aligned}$$

□

Corollary 2.2.18. *Let G be an unmixed bipartite graph with edge ideal I. Let $c = \operatorname{ht} I$. Let $t, \zeta_1, \ldots, \zeta_t, \widehat{\mathfrak{d}}$ be as in Discussion 2.2.6. Then*

$$\operatorname{depth} R/I = c - \max\left\{\sum_{i\in B} \zeta_i - |B| : B \text{ is an antichain of } \widehat{\mathfrak{d}}\right\}.$$

In particular, $\operatorname{depth} R/I \geq t$.

Proof. Let $\widehat{G}, S, \widehat{I}$ be as in Discussion 2.2.6. From Theorem 2.2.17, we know that if $\beta_{l,\sigma}((\widehat{I})^\star) \neq 0$ for some multidegree $\sigma \subseteq \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$, then $\sigma = \sigma_{A,B}$ for some antichains $B \subseteq A$ of $\widehat{\mathfrak{d}}$, with $|B| = l$. Now, in S, $\deg \sigma_{A,B} = \sum_{i \succcurlyeq A} \zeta_i + \sum_{i \not\succcurlyeq A} \zeta_i + \sum_{i \in B} \zeta_i = c + \sum_{i \in B} \zeta_i$. Hence

$$\operatorname{reg}(\widehat{I})^\star = c + \max\left\{ \sum_{i \in B} \zeta_i - |B| : B \text{ is an antichain of } \widehat{\mathfrak{d}} \right\}.$$

Note that $\operatorname{depth} R = \dim R = 2c$. Now apply Proposition 2.2.9, followed by the Auslander-Buchsbaum formula, to obtain the conclusion.

To show that $\operatorname{depth} R/I \geq t$, it suffices to show that, for all antichains B of $\widehat{\mathfrak{d}}$, $t + \sum_{i \in B} \zeta_i - |B| \leq c$. Since $c = \sum_{i=1}^{t} \zeta_i$, it suffices to show that $t - |B| \leq \sum_{i \notin B} \zeta_i$, which is true since $\zeta_i \geq 1$ for all i. □

Remark 2.2.19. The above bound is sharp. Given positive integers $t \leq c$, and a poset $\widehat{\mathfrak{d}}$ on t vertices, we can find an unmixed bipartite graph G on the vertex set $V = V_1 \bigsqcup V_2$ with edge ideal I such that $|V_1| = |V_2| = c$ and $\operatorname{depth} \Bbbk[V]/I = t$. Choose any antichain B in $\widehat{\mathfrak{d}}$ and set $\zeta_i = 1$ for all $i \notin B$. Choose $\zeta_i \geq 1, i \in B$ such that $\sum_{i \in B} \zeta_i = c - t + |B|$. Now construct a directed graph $\mathfrak{d}$ on c vertices by replacing the vertex i of $\widehat{\mathfrak{d}}$ by directed cycle of ζ_i vertices and then taking its transitive closure. Label the vertices of $\mathfrak{d}$ with $[c]$. Let G be a bipartite graph on $V = \{x_1, \ldots, x_c\} \bigsqcup \{y_1, \ldots, y_c\}$ such that $x_i y_i$ is an edge for all $i \in [c]$ and $x_i y_j$ is an edge whenever ij is a directed edge of $\mathfrak{d}$. Then G is an unmixed graph. We know from the corollary that $t \leq \operatorname{depth} R/I \leq c - \sum_{i \in B} \zeta_i - |B| = t$.

2.3 Examples

In this section, we discuss some examples where $\operatorname{reg} R/I > r(I)$.

	0	1	2	3	4	5
total	1	8	20	24	12	1
0	1	.	.	.	.	.
1	.	8	8	.	.	.
2	.	.	12	24	12	.
3	.	.	.	.	.	1

Table 2.1: Betti diagram for the 8-cycle

Example 2.3.1 (Non-Cohen-Macaulay bipartite graph). Let G be a cycle on 8 vertices with edge ideal

$$I = (x_1y_1, \dots, x_4y_4, x_1y_2, x_2y_3, x_3y_4, x_4y_1) \subseteq R = \Bbbk[x_1, \dots, x_4, y_1, \dots, y_4].$$

The Betti diagram of R/I is given in Table 2.1. Notice that $\operatorname{pd} R/I = 5$, while $I \subseteq (x_1, \dots, x_4)$ giving $\operatorname{ht} I \leq 4$. Hence G is not Cohen-Macaulay. From the Betti diagram, we see that $r(I) \leq 2$; in fact, x_1y_1 and x_2y_3 are disconnected, so $r(I) = 2$. However, $\operatorname{reg} R/I = 3$.

Example 2.3.2 (Cohen-Macaulay non-bipartite graph). Let Δ be the triangulation of the unit sphere $\mathbb{S}^2$ obtained by the regular icosahedron, represented in Figure 2.3. The facets of Δ are ABx, BCx, CDx, DEx, EAx, Acd, ABd, Bde, BCe, Cea, CDa, Dab, DEb, Ebc, EAc, aby, bcy, cdy, dey and eay, corresponding to the triangles in Figure 2.3. The Stanley-Reisner ideal I of Δ is generated by the pairs of vertices that do not form an edge in Figure 2.3;

$$\begin{aligned} I = (&ac, ad, bd, be, ce, aA, bA, eA, aB, bB, cB, bC, cC, dC, \\ &AC, cD, dD, eD, AD, BD, aE, dE, eE, BE, CE, ax, bx, cx, dx, \\ &ex, Ay, By, Cy, Dy, Ey, xy). \end{aligned}$$

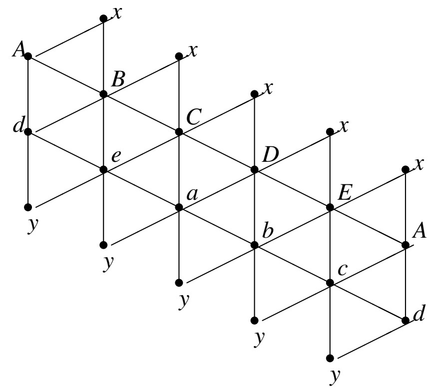


Figure 2.3: The net of a regular icosahedron

	0	1	2	3	4	5	6	7	8	9
total	1	36	160	327	412	412	327	160	36	1
0	1	.	.	.	.	.	.	.	.	.
1	.	36	160	315	300	112	12	.	.	.
2	.	.	.	12	112	300	315	160	36	.
3	.	.	.	.	.	.	.	.	.	1

Table 2.2: Betti diagram of a simplicial sphere

Thinking of this as the edge ideal of a graph G on the vertex set V, we see that G is Cohen-Macaulay; in fact, being the Stanley-Reisner ring of a simplicial sphere, R/I is Gorenstein. The Betti diagram of R/I is shown in Table 2.2. Indeed, if $\sigma \subseteq V$ has fewer than five elements, then $\widetilde{\mathrm{H}}_1(\Delta|_\sigma;\Bbbk) = 0$, so, by Hochster's formula (Discussion 1.3.7), we see that $\beta_{1,3} = \beta_{2,4} = 0$. From Proposition 2.2.2.b, we see that $r(I) = 1$. However, $\operatorname{reg}(R/I) = 3$.

2.4 Quasi-pure resolutions

In this section, we study Cohen-Macaulay bipartite graphs whose edge ideals have a quasi-pure resolution. Suppose that G is a graph on V with edge ideal I with the property that $\operatorname{reg} R/I = r(I)$. For instance, G is a forest (Proposition 2.2.2.a) or an unmixed bipartite graph (Theorem 2.2.15). If $r(I) \leq 2$, then R/I has a quasi-pure resolution. Now suppose that $r(I) \geq 3$ and that R/I has a quasi-pure resolution. Since $r(I) \geq 3$, $\overline{m}_3(I) = 6$, from Proposition 2.2.2.b. Hence $\underline{m}_4(I) \geq 6$. We claim that every vertex has at most 3 neighbours. More generally,

Proposition 2.4.1. *Let I be the edge ideal of a graph G. For any multidegree σ, $\beta_{|\sigma|-1,\sigma}(R/I) \neq 0$ if and only if there exists a partition $\sigma = \sigma_1 \bigsqcup \cdots \bigsqcup \sigma_d$ such that for all $1 \leq i \leq d$, for all $x \in \sigma_i$ and for all $y \in \sigma \setminus \sigma_i$, xy is an edge of $G|_\sigma$.*

Proof. We immediately reduce the problem to the case that $\sigma = V$, noting that, by Lemma 1.3.8(a) and the flatness of R over $\Bbbk[\sigma]$, $\beta_{|\sigma|-1,\sigma}(R/I) = \beta_{|\sigma|-1,\sigma}\left(\frac{R}{(I\cap\Bbbk[\sigma])R}\right) = \beta_{|\sigma|-1,\sigma}\left(\frac{\Bbbk[\sigma]}{(I\cap\Bbbk[\sigma])}\right)$ and that $(I \cap \Bbbk[\sigma])$ is the edge ideal of $G|_\sigma$ in the ring $\Bbbk[\sigma]$. Hence we need to show that $\beta_{|V|-1,V}(R/I) \neq 0$ if and only if there is a partition $V = V_1 \bigsqcup \cdots \bigsqcup V_d$ such that for all $1 \leq i \leq d$, for all $x \in V_i$ and for all $y \in V \setminus V_i$, xy is an edge of G. Let Δ be the coclique complex of G. Hochster's formula gives that $\beta_{|V|-1,V}(R/I) = \dim_\Bbbk \widetilde{\mathrm{H}}_0(\Delta;\Bbbk)$. Hence we must show that Δ is disconnected if and only if a partition, such as above, exists.

Suppose such a partition exists. Then any coclique of G is contained in V_i for some $1 \leq i \leq d$; hence Δ is disconnected. Conversely, assume that Δ is disconnected. Denoting the number of distinct components of Δ by d, we set $V_i, 1 \leq i \leq d$, be the vertex sets of these components. We see immediately that for $x, y \in V$, whenever x and y are in different components of Δ, there is an edge xy in G. □

We wish to mention here that this agrees with the result of Novik-Swartz [NS06, Theorem 1.3] that the first skip in the sequence of $\underline{m}_l$'s is at $n - q_1 + 1$, where q_1 is the Cohen-Macaulay connectivity of the 1-dimensional skeleton of the Stanley-Reisner complex of I. For the edge ideal of a graph G, the 1-dimensional skeleton of its Stanley-

Reisner complex is the complement graph $\bar{G}$. In passing, let us note that if G is a forest, then Proposition 2.4.1 implies that $\max\{l : \underline{m}_l(I) = l+1\} = \max\{\deg_G x : x \in V\}$. More generally, if G is a bipartite graph, then $\max\{l : \underline{m}_l(I) = l+1\}$ is the largest cardinality of a complete bipartite subgraph of G.

Discussion 2.4.2. We already observed that if $\operatorname{reg} R/I \leq 2$, then R/I has a quasi-pure resolution. Let G be a connected Cohen-Macaulay bipartite graph such that $\operatorname{reg} R/I \geq 3$ and R/I has a quasi-pure resolution. It is easy to see that G is connected if and only if $\mathfrak{d}_G$ is a connected poset. Since $\operatorname{reg} R/I = r(I) \geq 3$, $\overline{m}_3(I) = 6$ by Proposition 2.2.2.b, so for R/I to have a quasi-pure resolution, we must have $\underline{m}_4(I) \geq 6$. This means, by the observation in the last paragraph, that for all i, there are at most two elements j such that $j \succcurlyeq i$ (or $i \succcurlyeq j$) in $\mathfrak{d}_G$. Since $\mathfrak{d}_G$ is connected, in every maximal chain, there exists i, j, j' such that j and j' cover i or i covers j and j'. From the observation above, it follows that, in the first case, i is a source vertex and that j and j' are sink vertices. Similarly, in the second case, i is a sink vertex and j and j' are source vertices. Hence every maximal chain of $\mathfrak{d}_G$ has length at most one; in fact, since $\mathfrak{d}_G$ is connected, every maximal chain has length one. Therefore every vertex in $\mathfrak{d}_G$ is a source vertex or a sink vertex, but not both. Every source (respectively, sink) vertex in $\mathfrak{d}_G$ is covered by (respectively, covers) at most two sink (respectively, source) vertices. For x_iy_i to be a leaf in G, it is necessary and sufficient that i is a source vertex or a sink vertex in $\mathfrak{d}_G$. Therefore, in our case, x_iy_i is a leaf for all i; in other words, G is the suspension of its subgraph G' induced on the set of vertices $V' := \{x_i : i \text{ is a source vertex of } \mathfrak{d}_G\} \cup \{y_i : i \text{ is a sink vertex of } \mathfrak{d}_G\} \subseteq V$. The underlying undirected graph $|\mathfrak{d}_G|$ is, first, bipartite, and, secondly, a path on c vertices (necessarily, if c is odd) or a cycle on c vertices. The subgraph G' of G described above is a path (respectively, a cycle) if and only if $|\mathfrak{d}_G|$ is a path (respectively, a cycle).

Lemma 2.4.3. *Let G be a Cohen-Macaulay bipartite graph. Then G is a forest if and only if the undirected graph $|\mathfrak{d}_G|$ has no cycles.*

Proof. Let $i_1, i_2, \ldots, i_t, i_{t+1} = i_1$ be a cycle in $|\mathfrak{d}_G|$. Set $C_0 = \varnothing$ and define, inductively, for $l = 1, \ldots, t$,

$$C_l = C_{l-1} \bigcup \begin{cases} x_{i_l} y_{i_{l+1}}, & \text{if } i_{l+1} \succ i_l, \\ y_{i_l} x_{i_{l+1}}, & \text{otherwise.} \end{cases}$$

It is easy to see that there is a cycle among the edges $C_t \cup \{x_i y_i, i = i_1, \ldots, i_t\}$.

Conversely, if G has a cycle, then let C be the edges $x_i y_j$ in the cycle with $i \neq j$. Then we note that the edges ij such that $x_i y_j \in C$ form a cycle in $|\mathfrak{d}_G|$. □

Remark 2.4.4. Then every chain in $\mathfrak{d}_G$ has length at most one. For, if $\mathfrak{d}_G$ had a chain of length two, labelled, say, $1 \prec 2 \prec 3$, then $x_1y_2, x_2y_2, x_2y_3, x_1y_3$ form a cycle in G. Conversely, any cycle in G gives rise to a chain of length at least two in $\mathfrak{d}_G$.

Proposition 2.4.5. *Let I be the edge ideal of a connected Cohen-Macaulay bipartite graph G. Then the following are equivalent:*

a. $\operatorname{reg} R/I \geq 3$ *and R/I has a quasi-pure resolution.*

b. G is the suspension of the path on five or six vertices or of the 6*-cycle.*

Proof. (a) $\Longrightarrow$ (b): First, if $c \geq 7$, then we claim that R/I cannot have a quasi-pure resolution. Since $\operatorname{reg} R/I \geq 3$, $\mathfrak{d}_G$ is such that every vertex is a source vertex or a sink vertex, but not both, and that every source (respectively, sink) vertex in $\mathfrak{d}_G$ is covered by (respectively, covers) at most two sink (respectively, source) vertices. If $c > 7$, then restrict $\mathfrak{d}_G$ to one of its connected subgraphs with seven vertices. This corresponds to restricting G to a Cohen-Macaulay subtree on 14 vertices. If we show that the edge ideal of this subtree does not have a quasi-pure resolution, then, by Lemma 1.3.8(a), we have that R/I does not have a quasi-pure resolution. Therefore, by replacing G by this subgraph, we may assume that G is a Cohen-Macaulay tree on 14 vertices, such that the length of every maximal path in $\mathfrak{d}_G$ is one, and prove the R/I does not have a quasi-pure resolution. We may verify this with a computer algebra system, such as [M2], but we give a direct proof below.

We will prove this when $\mathfrak{d}_G$ has four source vertices and three sink vertices. The other case is of $\mathfrak{d}_G$ with three source vertices and four sink vertices; this corresponds to relabelling the partition of the vertex set. Since $c = 7$ is odd, G is the suspension of a path on 7 vertices. We label the source vertices $1, \cdots, 4$ and the sink vertices $5, 6, 7$. Then the edges of $\mathfrak{d}_G$ are $15, 25, 26, 36, 37$ and 47. Hence $I = (x_1y_1, \cdots, x_7y_7, x_1y_5, x_2y_5, x_2y_6, x_3y_6, x_3y_7, x_4y_7)$. We saw that $m_4(I) = 6$. Since the set of four source vertices in $\mathfrak{d}_G$ form an antichain, $\operatorname{reg} R/I = 4$, and hence $M_4(I) = 8$; to prove that R/I does not have a quasi-pure resolution, we need to show that $m_5(I) \leq 7$. Let $\sigma = \{x_1, y_5, x_2, y_6, x_3, y_7, x_5\}$, and $J = (I \cap \Bbbk[\sigma])R = (x_1y_5, x_2y_5, x_5y_5, x_2y_6, x_3y_6, x_3y_7)$. We will show that $\beta_{5,7}(R/J) \neq 0$, which will suffice, by Lemma 1.3.8(a), to show that $m_5(I) \leq 7$. We have a short exact sequence of graded R-modules

$$0 \longrightarrow \frac{R}{(J:y_5)}(-1)) \longrightarrow R/J \longrightarrow R/(J, y_5) \longrightarrow 0.$$

Since $R/(J, y_5)$ is Cohen-Macaulay and $\operatorname{ht}(J, y_5) = 3$, we see from the associated long exact sequence of $\operatorname{Tor}(\Bbbk, -)$ that

$$\operatorname{Tor}_5(\Bbbk, \frac{R}{(J:y_5)}(-1)) \simeq \operatorname{Tor}_5(\Bbbk, R/J).$$

To complete the argument, we will show that $\beta_{5,6}(R/(J:y_5)) \neq 0$. Since $(J:y_5) = (x_1, y_2, x_3, x_4x_5, x_5x_6)$, this is equivalent to $\beta_{2,3}(R/(x_4x_5, x_5x_6)) \neq 0$, which is true.

We showed so far that $c \leq 6$. Now, if $c < 5$, $\operatorname{reg} R/I < 3$. Hence $c = 5$ or $c = 6$. As we noted in Discussion 2.4.2 that G, therefore, is the suspension of the path or the cycle in five or six vertices or of the 6-cycle.

(b) $\implies$ (a): If G is the suspension of the path or the cycle on c vertices, then $\mathfrak{d}_G$ is such that every vertex is a source vertex or a sink vertex, but not both, and that every source (respectively, sink) vertex in $\mathfrak{d}_G$ is covered by (respectively, covers) at most two sink (respectively, source) vertices. Hence $\operatorname{reg} R/I = \lceil \frac{c}{2} \rceil$. Since $c = 5$ or $c = 6$ in our case, $\operatorname{reg} R/I = 3$. With this, R/I has a quasi-pure resolution if and only if $m_4(I) = 6$, which we now show. If on the other hand, $m_4(I) = 5$, then there exists $\sigma \subseteq V$ and a partition $\sigma = \sigma_1 \bigsqcup \sigma_2$ (into two sets, since G is bipartite) such that $|\sigma| = 5$ and $G|_\sigma$ is a complete bipartite graph (Proposition 2.4.1). Recall that $V = V_1 \bigsqcup V_2$ is the partition of the vertex set V of G. We may assume that $\sigma_i \subseteq V_i, i = 1,2$. If $|\sigma_i| = 1$ for any i, then $|\sigma| \leq 4$, because $\deg_G x \leq 3$ for all $x \in V$. On the other hand, if, say, $|\sigma_1| \geq 2$, then $|\sigma_2| = 1$, because otherwise, we would get a 4-cycle in G, contradicting the fact that G has only a 6-cycle, if any. Now, again, $|\sigma_1| \leq 3$, so $|\sigma| < 5$. Hence R/I has a quasi-pure resolution. $\square$

We add, in passing, that the edge ideals I of the suspension of paths and cycles on four or fewer vertices have quasi-pure resolutions, but this follows easily from the fact that $\operatorname{reg} R/I \leq 2$.

2.5 Discussion

Many classes of simplicial complexes have Stanley-Reisner ideals that are generated by quadratic monomials. Further, homogeneous ideals that have quadratic initial ideals (in some monomial order) define Koszul algebras, which have important homological properties. Therefore it is worth while to understand the resolutions of quadratic monomial ideals (equivalently, of square-free quadratic monomial ideals, or edge ideals of graphs).

At the outset, we may wish to strengthen Proposition 2.2.2.b, by finding an invariant of a graph G that gives $\operatorname{reg} R/I$. From the discussion immediately preceding Proposition 2.2.2, we note that the invariant $r(I)$ captures Koszul-like syzygies among the generators of I. One of the future problems is to determine whether there are (induced) subgraphs G that determine $\operatorname{reg} R/I$. Similarly, characterizing Cohen-Macaulay graphs with pure or quasi-pure resolutions is also open.

Generalizing Proposition 2.2.9, Theorem 2.2.15 and Corollary 2.2.18 to the class of all bipartite graphs is also an important future direction.

Chapter 3

Arithmetic Rank of Bipartite Edge Ideals

In this chapter, we prove that for certain Cohen-Macaulay graphs, the edge ideal can be generated up to radical by as many polynomials as its height. In other words, the arithmetic rank of the edge ideal is its height.

3.1 Arithmetic Rank

Let S be any commutative Noetherian ring, and $I \subseteq S$ an ideal. The *arithmetic rank* of I, denoted $\operatorname{ara} I$, is the least number t such that there exists $f_1, \ldots, f_t \in S$ such that $\sqrt{(f_1, \ldots, f_t)} = \sqrt{I}$. Now if $J \subseteq S$ is an ideal such that $\sqrt{I} = \sqrt{J}$, then, for all $\mathfrak{p} \in \operatorname{Spec} S$, $I \subseteq \mathfrak{p}$ if and only if $J \subseteq \mathfrak{p}$. It follows immediately from the Krull principal ideal theorem [Eis95, Theorem 10.2] that $\operatorname{ara} I \geq \operatorname{ht} I$.

Let V be a finite set of indeterminates over a field $\Bbbk$ and let $R = \Bbbk[V]$. For an ideal $I \subseteq R$, we say that R/I is a *set-theoretic complete intersection* if $\operatorname{ara} I = \operatorname{ht} I$. This terminology arises as follows: let $c = \operatorname{ht} I$. If $\sqrt{I} = \sqrt{(f_1, \ldots, f_c)}$ for some $f_1, \ldots, f_c \in R$, then $\operatorname{Spec} R/I$ is homeomorphic to $\operatorname{Spec} R/(f_1, \ldots, f_c)$. Observe that $\operatorname{ht}(f_1, \ldots, f_c) = c$, *i.e.*, the images of $f_1, \ldots, f_c$ form part of a system of parameters in $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $(f_1, \ldots, f_c) \subseteq \mathfrak{p}$. Since $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec} R$, the images of the f_j form a regular sequence in $R_{\mathfrak{p}}$, for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $(f_1, \ldots, f_c) \subseteq \mathfrak{p}$. Therefore $\operatorname{Spec} R/(f_1, \ldots, f_c)$ is a complete intersection.

Arithmetic rank of an ideal $I \subseteq R$ is related to the vanishing of local cohomology of R-modules with support in I. For a discussion of this, see [Hun07, Section 5] or [ILL$^+$07, Section 9.3].

From now on, we assume that $I \subseteq R$ is a square-free monomial ideal. Arithmetic rank of monomial ideals has been considered by P. Schenzel and W. Vogel [SV77], T. Schmitt and Vogel [SV79] and G. Lyubeznik [Lyu88b]. Lyubeznik showed that for a square-free monomial ideal I, $\operatorname{pd} R/I \leq \operatorname{ara} I \leq \max_t \operatorname{pd} R/I^{(t)}$; see [Lyu88b, Proposition 3]. Here $I^{(t)} := \bigcap_{\mathfrak{p} \in \operatorname{Ass} R/I} \mathfrak{p}^t$ is the tth *symbolic power* of I. This result is similar to — and its proof indeed uses — a theorem of L. Burch [Bur72, Corollary, p. 373] that the analytic spread of an ideal J of a local Noetherian ring A is at most $\dim A - \min_t \operatorname{depth} A/J^t$. It is known, however, due to Z. Yan [Yan00a, Example 2] that, in general, $\operatorname{pd} R/I$ and $\operatorname{ara} I$ need not be equal. If they are equal, and R/I is Cohen-Macaulay, then it would mean that R/I is a set-theoretic complete intersection. We add, parenthetically, that if R/I is a set-theoretic complete intersection, then, from the result of Lyubeznik mentioned above, $\operatorname{ht} I = \operatorname{pd} R/I = \operatorname{ara} I$, *i.e.*, R/I is Cohen-Macaulay. In the next section, we will describe a class of Cohen-Macaulay bipartite graphs G for which $\operatorname{ht} I = \operatorname{pd} R/I = \operatorname{ara} I$. In Section 3.3, we will see some examples of Cohen-Macaulay bipartite graphs that belong to the above class. Upper bounds for arithmetic rank have also been considered by M. Barile [Bar96] and [Bar06], building on the work of Schmitt and Vogel mentioned above.

3.2 Main Result

On $\mathbb{N}^2$, we define a poset by setting $(a,b) \geq (c,d)$ if $a \geq c$ and $b \geq d$. (A general reference for results on posets is [Sta97, Chapter 3].) Let $(P_1, \geq_1)$, $(P_2, \geq_2)$ be two finite posets, respectively, on vertex sets W_1 and W_2. We say that P_1 is *isomorphic* to P_2 if there is a bijection $\phi : W_1 \longrightarrow W_2$ such that for all $i, j \in W_1$, $j \geq_1 i$ if and only if $\phi(j) \geq_2 \phi(i)$. We say that P_1 can be *embedded isomorphically* in $\mathbb{N}^2$ if there exists a map $\phi : W_1 \longrightarrow \mathbb{N}^2$ such that all $i, j \in W_1$, $j \geq_1 i$ if and only if $\phi(j) \geq \phi(i)$; such a map ϕ will be called an *embedding* of P_1 in $\mathbb{N}^2$. We will denote the projection of $\mathbb{N}^2$ along the first co-ordinate by π. The main result of this chapter is,

Theorem 3.2.1. *Let G be a Cohen-Macaulay bipartite graph with edge ideal I. If $\mathfrak{d}_G$ has an embedding in $\mathbb{N}^2$, then* $\operatorname{ara} I = \operatorname{ht} I$.

Definition 3.2.2. *Let $(P, \succcurlyeq)$ be a finite poset on a finite vertex set W, with an embedding ϕ in $\mathbb{N}^2$. Then there is a unique $i_0 \in W$ such that i_0 is minimal in P and $(\pi \circ \phi)(i_0)$ is minimum. Similarly, let j_0 be the unique maximal element such that $(\pi \circ \phi)(j_0)$ is minimum. Let P_1 and P_2 be the restrictions of P respectively to $W \setminus \{i_0\}$ and $W \setminus \{j_0\}$. The* column linearization *of P induced by ϕ is the map $\gamma : W \longrightarrow [|W|]$ defined recursively as follows:*

$$\gamma(i) = \begin{cases} 1, & i = i_0 \\ 1 + \gamma_1(i), & i \neq i_0 \end{cases}$$

where γ_1 is a column linearization of P_1 induced by ϕ. A row linearization *of P induced by ϕ is the map $\rho : W \longrightarrow [|W|]$ defined recursively as follows:*

$$\rho(j) = \begin{cases} 1, & j = j_0 \\ 1 + \rho_1(j), & j \neq j_0 \end{cases}$$

where ρ_1 is a row linearization of P_2 induced by ϕ. We will say that (γ, ρ) is the pair of linearizations induced by ϕ. $\square$

Proposition 3.2.3. *Let P, ϕ, γ and ρ be as in Definition 3.2.2. For $i, j \in P$, if $j \succcurlyeq i, j \neq i$, then $\gamma(j) > \gamma(i)$ and $\rho(j) < \rho(i)$. If i and j are incomparable, then $\gamma(j) > \gamma(i)$ if and only if $\rho(j) > \rho(i)$.*

Proof. If $j \succcurlyeq i$, then $\phi(j) \geq \phi(i)$. In the recursive definition of γ, i would appear as the unique minimal vertex with the smallest value of $(\pi \circ \phi)$ before j would, so $\gamma(i) < \gamma(j)$. On the other hand, while computing ρ recursively, j would appear as the unique maximal vertex with the smallest value of $(\pi \circ \phi)$ before i would, so $\rho(j) < \rho(i)$. Therefore $\tilde{\phi}(j) \geq \tilde{\phi}(i)$. On the other hand, if i and j are incomparable, then we may assume without loss of generality that $(\pi \circ \phi)(i) < (\pi \circ \phi)(j)$. Hence, while computing γ and ρ recursively, i will be chosen before j, giving $\gamma(i) < \gamma(j)$ and $\rho(i) < \rho(j)$, which implies that $\tilde{\phi}(j)$ and $\tilde{\phi}(i)$ are incomparable. $\square$

Discussion 3.2.4. Let P be a poset on a finite set W, with an embedding ϕ in $\mathbb{N}^2$. Let (γ, ρ) be the pair of linearizations of ϕ induced by ϕ. Let $E = \{(\gamma(i), \rho(j)) : j \succcurlyeq i \in W\} \subseteq \mathbb{R}^2$. We think of E as a subset of $[|W|] \times [|W|]$ in the first quadrant of the Cartesian plane. Let i, j be such that $(\gamma(i), \rho(j)) \in E$ is not the lowest vertex in its column, *i.e.*, there exists l such that $(\gamma(i), \rho(l))$ lies below $(\gamma(i), \rho(j))$. Then $j \succcurlyeq i$, $l \succcurlyeq i$ and, from Proposition 3.2.3, $l \neq i$. Therefore, again from Proposition 3.2.3, $\gamma(l) > \gamma(i)$ and $(\gamma(i), \rho(l))$ is not the right-most vertex in its row. Let k be such that $(\gamma(k), \rho(l))$ lies immediately to the right of $(\gamma(i), \rho(l))$ in its row. Draw an edge between $(\gamma(i), \rho(j))$ and $(\gamma(k), \rho(l))$. Repeating this for all $j \succcurlyeq i$ such that $(\gamma(i), \rho(j))$ is not the lowest vertex in its column, we obtain a graph Γ on W. Rows and columns of Γ will be indexed starting from the bottom left corner.

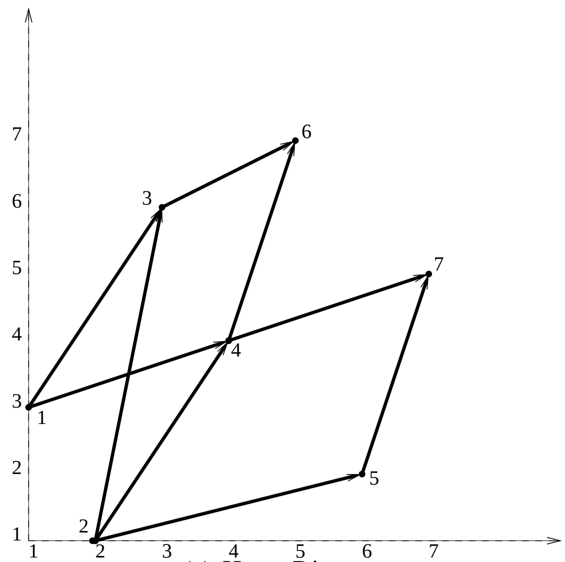
Example 3.2.5. Let P be a poset on a vertex set W. The *Hasse diagram* of P is a directed graph H on W obtained by setting ij to be an edge (directed from i to j) if j covers i (see Discussion 2.2.1). Let G be a Cohen-Macaulay bipartite graph on the vertex set $\{x_1, y_1, \ldots, x_7, y_7\}$ whose associated directed graph $\mathfrak{d}_G$ has the Hasse diagram in Figure 3.1(a), on p. 62. We have drawn an embedding of the Hasse diagram in $\mathbb{N}^2$, so that we can read off the functions γ and ρ without much effort: $\gamma(1) = 1, \gamma(2) = 2, \gamma(3) = 3, \gamma(4) = 4, \gamma(6) = 5, \gamma(5) = 6, \gamma(7) = 7$; $\rho(6) = 1, \rho(3) = 2, \rho(7) = 3, \rho(4) = 4, \rho(1) = 5, \rho(5) = 6, \rho(2) = 7$. Conventionally, we would omit the coordinate system while drawing Hasse diagrams. In Figure 3.1(b), the graph Γ described in Discussion 3.2.4 is given. Notice that γ determines the order of the x_i while ρ determines the order of the y_j.

Lemma 3.2.6. *With notation as in Discussion 3.2.4, Γ has exactly $|W|$ connected components.*

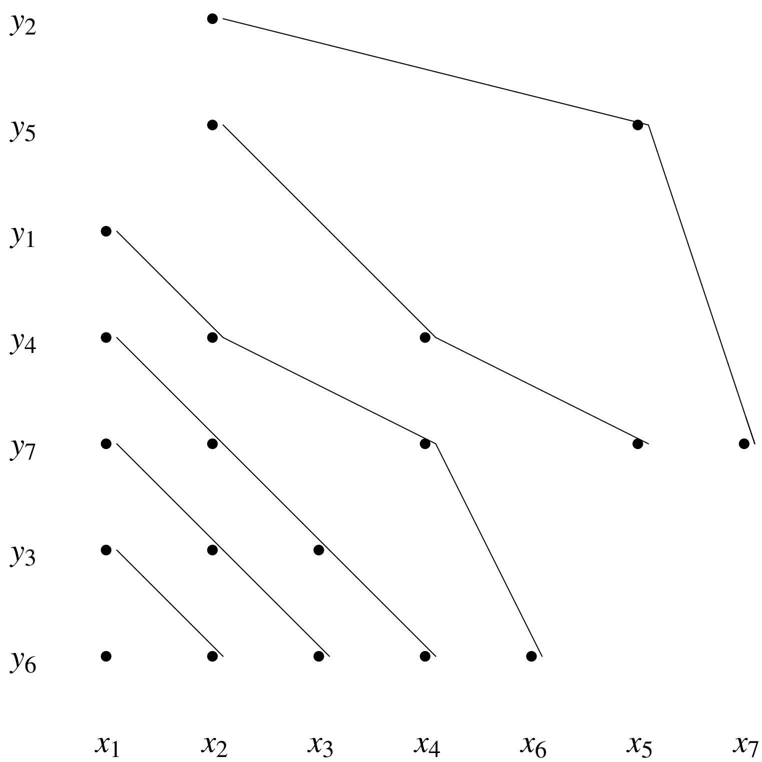
Proof. Suppose that C is a connected component of Γ and that $(\gamma(i), \rho(j))$ is the top left vertex of C. We claim that it is the left-most vertex in its row. For, if not, then there exists k such that $(\gamma(k), \rho(j))$ lies immediately to the left of $(\gamma(i), \rho(j))$. From Proposition 3.2.3, $k \neq j$. We note, again from Proposition 3.2.3, that $(\gamma(k), \rho(j))$ is not the top-most vertex in its column, contradicting the hypothesis that that $(\gamma(i), \rho(j))$ is the top left vertex of C. Now, there are exactly $|W|$ rows in Γ. $\square$

Lemma 3.2.7. *Let G be a Cohen-Macaulay bipartite graph such that ϕ is an embedding of $\mathfrak{d}_G$ in $\mathbb{N}^2$. Let (γ, ρ) be the pair of linearizations induced by ϕ. Then the vertices in the first column of Γ belong to a contiguous set of rows, starting with row* 1.

Proof. We may assume that the labelling of $\mathfrak{d}_G$ is such that $\gamma^{-1}(1) = 1$ and $\gamma^{-1}(2) = 2$. We need to show that $\rho(i) > \rho(1)$ if $i \not\geq 1$. Proposition 3.2.3 gives that 1 is minimal



(a) Hasse Diagram



(b) The Graph Γ

Figure 3.1: Example 3.2.5

in $\mathfrak{d}_G$. Let $i \not\succcurlyeq 1$. Then i and 1 are incomparable. Since $\gamma(1) = 1 \leq \gamma(i)$, we see, again from Proposition 3.2.3, that $\rho(i) > \rho(1)$. $\square$

Remark 3.2.8. Let P be a poset on a finite vertex set W with an embedding ϕ in $\mathbb{N}^2$. Let (γ, ρ) be the pair of linearizations induced by ϕ. Let $W' = W \setminus \{\gamma^{-1}(1)\}$ and let P' be the restriction of P to W'. Then $\phi|_{W'}$ is an embedding of P in $\mathbb{N}^2$. For $i \in W'$, set $\gamma'(i) = \gamma(i) - 1$, and

$$\rho'(i) = \begin{cases} \rho(i), & i \succcurlyeq \gamma^{-1}(1) \\ \rho(i) - 1, & \text{otherwise.} \end{cases}$$

Then (γ', ρ') is the pair of linearizations induced by $\phi|_{W'}$. Let Γ' be the graph constructed from P' as described in Discussion 3.2.4 using γ' and ρ'. Then Γ' is obtained by deleting the vertices in the first column of Γ. We see this as follows. For all $i, j \in W'$, $\rho(i) < \rho(j)$ if and only if $\rho'(i) < \rho'(j)$; similarly, $\gamma(i) < \gamma(j)$ if and only if $\gamma'(i) < \gamma'(j)$. Further, there is only one vertex in row $\rho(\gamma^{-1}(1))$ in Γ, and this is in the first column.

Remark 3.2.9. Let P be a poset on a finite vertex set W with an embedding ϕ in $\mathbb{N}^2$. Let (γ, ρ) be the pair of linearizations induced by ϕ. Let $W' = W \setminus \{j \subseteq \gamma^{-1}(1)\}$ and let P' be the restriction of P to W'. Then $\phi|_{W'}$ is an embedding of P in $\mathbb{N}^2$. Let $\tilde{\gamma}$ be the order-preserving map from $\operatorname{Im} \gamma|_{W'}$ to $[|W'|]$. Let $\gamma' := \tilde{\gamma} \circ \gamma|_{W'}$. For $j \in W'$, set $\rho'(j) = \rho(j) - \rho(1)$. Then (γ', ρ') is the pair of linearizations of P' induced by $\phi|_{W'}$. Let Γ' be the graph constructed from P' as described in Discussion 3.2.4 using $\tilde{\gamma} \circ \gamma|_{W'}$ and $\tilde{\rho} \circ \rho|_{W'}$. We claim that Γ' is the graph obtained from Γ by deleting the vertices that lie in rows $\rho(j)$ for some $j \succcurlyeq \gamma^{-1}(1)$. For, first observe that for all $i, j \in W'$, $\rho(i) < \rho(j)$ if and only if $\rho'(i) < \rho'(j)$; similarly, $\gamma(i) < \gamma(j)$ if and only if $\gamma'(i) < \gamma'(j)$. Moreover, for all $j \succcurlyeq \gamma^{-1}(1)$, the vertices in the column $\gamma(j)$ belong to rows between 1 and $\rho(j)$ (possibly, not all of them). Therefore, after the vertices in the rows between 1 and $\rho(1)$ have been deleted, the remaining vertices belong to columns $\gamma(j)$ for $j \not\succcurlyeq 1$. Hence

$(\gamma'(i), \rho'(j))$ and $(\gamma'(k), \rho'(l))$ belong to the same connected component of Γ' if and only if $(\gamma(i), \rho(j))$ and $(\gamma(k), \rho(l))$ belong to the same connected component of Γ.

Before we give a proof of Theorem 3.2.1, we illustrate the arguments for the graph in Example 3.2.5. Let I be the edge ideal of G. Define

$$g_1 = x_1y_6, \qquad g_2 = x_2y_6 + x_1y_3, \qquad g_3 = x_3y_6 + x_2y_3 + x_1y_7,$$
$$g_4 = x_4y_6 + x_3y_3 + x_2y_7 + x_1y_4, \qquad g_5 = x_6y_6 + x_4y_7 + x_2y_4 + x_1y_1,$$
$$g_6 = x_5y_7 + x_4y_4 + x_2y_5, \qquad g_7 = x_7y_7 + x_5y_5 + x_2y_2.$$

Note that each form above is obtained by taking the sum of the monomials corresponding to vertices in a connected component of Γ; see Figure 3.1(b). Let $J = (g_1, \ldots, g_7)$. Using Remarks 3.2.8 and 3.2.9, we may assume that $\sqrt{(J, x_1)} = (I, x_1)$ and that $\sqrt{(J : x_1)} = (I : x_1)$. Hence for all $\mathfrak{p} \in \operatorname{Spec} R$, $J \subseteq \mathfrak{p}$ if and only if $I \subseteq \mathfrak{p}$, giving that $\sqrt{J} = I$. We are now ready to prove the theorem.

Theorem 3.2.1. *Let G be a Cohen-Macaulay bipartite graph with edge ideal I. If $\mathfrak{d}_G$ has an embedding in $\mathbb{N}^2$, then* $\operatorname{ara} I = \operatorname{ht} I$.

Proof. Denote the embedding of $\mathfrak{d}_G$ by ϕ, and let (γ, ρ) be pair of linearizations induced by ϕ. Let Γ be the graph constructed as in Discussion 3.2.4. We prove the theorem by induction on c. Since the conclusion is evident when $c = 1$, we assume that $c > 1$ and that it holds for all Cohen-Macaulay bipartite graph on fewer than $2c$ vertices. For $t = 1, \ldots, c$, let C_t be the connected component of Γ containing the left most vertex in row t. We saw in the proof of Lemma 3.2.7 that these are exactly the connected components of Γ. Set

$$g_t = \sum_{(\gamma(i),\rho(j))\in C_t} x_i y_j \qquad 1 \le t \le c.$$

Set $J = (g_1, \ldots, g_c)$. We will show that $I = \sqrt{J}$, or, equivalently, that for all $\mathfrak{p} \in \operatorname{Spec} R$, $I \subseteq \mathfrak{p}$ if and only if $J \subseteq \mathfrak{p}$. Further, without loss of generality, we may assume that $\gamma^{-1}(1) = 1$. Then 1 is a minimal element of $\mathfrak{d}_G$. Let $W_1 := \{2, \ldots, c\}$ and $W_2 := \{i \not\succcurlyeq 1\} \subseteq [c]$. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ respectively be the restrictions of $\mathfrak{d}_G$ to W_1 and W_2.

Let G_1 be the deletion of x_1 and y_1 in G, whose edge ideal (in $R = \Bbbk[V]$) is $((I, x_1) \cap \Bbbk[x_2, y_2, \ldots, x_c, y_c])R$. Note that $\mathfrak{d}_1$ is the associated directed graph of G_1. Let Γ_1 denote the deletion of the vertices that lie in the first column of Γ. Write $J_1 = ((J, x_1) \cap \Bbbk[x_2, y_2, \ldots, x_c, y_c])R$. We see from Remark 3.2.8 that that J_1 is defined from Γ_1 precisely the same way that J is defined from Γ. Along with the induction hypothesis, this gives that $((I, x_1) \cap \Bbbk[x_2, y_2, \ldots, x_c, y_c])R = \sqrt{J_1}$. Note that $(J_1, x_1) = (J, x_1)$, so we obtain that $(I, x_1) = \sqrt{(J, x_1)}$. We thus see that for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $x_1 \in \mathfrak{p}$, $I \subseteq \mathfrak{p}$ if and only if $J \subseteq \mathfrak{p}$.

Let G_2 be the deletion of x_1 and all its neighbours in G; its edge ideal is $((I : x_1) \cap \Bbbk[x_i, y_i : i \in W_2])R$. The associated directed graph of G_2 is $\mathfrak{d}_2$. Let Γ_2 denote the deletion of the vertices that lie columns $\gamma(i)$ or in rows $\rho(i)$ of Γ whenever $i \succcurlyeq 1$. Let

$$J_2 = ((J + (y_i : i \succcurlyeq 1)) \cap \Bbbk[x_i, y_i : i \not\succcurlyeq 1])R.$$

From Remark 3.2.8, we note that J_2 is defined from Γ_2 precisely the same way that J is defined from Γ. This, along with the induction hypothesis, implies that $((I : x_1) \cap \Bbbk[x_i, y_i : i \in W_2])R = \sqrt{J_2}$. Now, $J_2 + (y_i : i \succcurlyeq 1) = J + (y_i : i \succcurlyeq 1) = (J : x_1)$, so $(I :$

$x_1) = \sqrt{(J : x_1)}$. We thus see that for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $x_1 \notin \mathfrak{p}$, $I \subseteq \mathfrak{p}$ if and only if $J \subseteq \mathfrak{p}$. Together with the previous paragraph, we conclude that $\sqrt{J} = I$. $\square$

3.3 Examples

We will first see some examples and non-examples of Cohen-Macaulay bipartite graphs that satisfy the hypothesis of Theorem 3.2.1.

Example 3.3.1 (Cohen-Macaulay Trees - Example). Recall that a *tree* is a connected acyclic graph. Acyclic graphs are bipartite. A Cohen-Macaulay bipartite graph G is acyclic if and only if every maximal chain in $\mathfrak{d}_G$ has length exactly 1 and the underlying undirected graph of $\mathfrak{d}_G$ (which is a poset) has no cycles (Lemma 2.4.3). We show an example of such a poset that can be embedded in $\mathbb{N}^2$, and of one that can not be embedded. Let $n \in \mathbb{N}$ and P be the poset on $[n]$ such that even numbers are maximal, odd numbers minimal, and for all $i, j \in [n]$, i and j are incomparable unless $|i - j| = 1$. Then, $\phi : [n] \to \mathbb{N}^2$ defined below is an embedding of P in $\mathbb{N}^2$. Let $i \in [n]$ and $m \gg 0 \in \mathbb{N}$.

$$\phi(i) = \begin{cases} \left(\frac{i+1}{2}, m - \frac{i+1}{2}\right), & i \text{ odd}, \\ \left(\frac{i}{2} + 1, m + 1 - \frac{i}{2}\right), & i \text{ even}. \end{cases}$$

Example 3.3.2 (Cohen-Macaulay Trees - Non-example). Let P be the poset on 7 vertices shown in Figure 3.2 Suppose, by way of contradiction, that ϕ is an embedding in $\mathbb{N}^2$. Let $\phi(i) = (a_i, b_i), 1 \leq i \leq 7$. Then there exist $i \neq j \in \{1,3,4\}$ such that $(a_i - a_2)(a_j - a_2) > 0$. Hence we may assume that $a_2 < a_3 < a_4$. Then $b_2 > b_3 > b_4$. Since ϕ is an embedding, $a_7 > a_4$ and $b_7 > b_2$, giving $\phi(7) > \phi(3)$. However, $7 \not> 3$ in P.

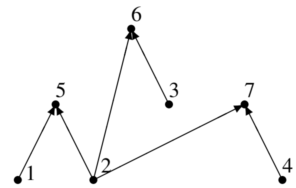


Figure 3.2: Example 3.3.2

Remark 3.3.3. Every finite poset can be embedded in a finite Boolean lattice, and every finite Boolean lattice can be embedded in a finite product of copies of $\mathbb{N}$. If P is a poset on a finite vertex set W, then the map $i \mapsto \{j \not\geq i\}$ is an embedding of P in $\mathscr{B}_{|W|}$. For $n \in \mathbb{N}$, the Boolean lattice $\mathscr{B}_n$ is isomorphic to $\{0,1\}^n$, with the order $(b_1, \ldots, b_n) \geq (a_1, \ldots, a_n)$ if and only if $b_i \geq a_i$ for all $1 \leq i \leq n$. The latter poset can clearly be embedded in $\mathbb{N}^n$.

3.4 Further Questions

The most obvious question, which this chapter does not answer, is whether the edge ideal I of a Cohen-Macaulay bipartite graph G whose associated directed graph $\mathfrak{d}_G$ cannot be embedded in $\mathbb{N}^2$ defines a set-theoretic complete intersection. More generally,

Question 3.4.1. Let I be a square-free monomial ideal. Under what conditions is $\operatorname{ara} I = \operatorname{pd} R/I$?

In the example of Z. Yan [Yan00a, Example 2], projective dimension of the ideal depends on the characteristic of the field $\Bbbk$, while arithmetic rank does not. This raises the following questions:

Question 3.4.2. Let I be a square-free monomial ideal. Is $\operatorname{ara} I$ independent of the characteristic of $\mathbb{k}$? Moreover, is it equal to the maximum value that $\operatorname{pd} R/I$ takes, as the characteristic changes over the set of prime integers?

Chapter 4

Multiplicity Bounds for Quadratic Monomial Ideals

In this chapter, we study a series of conjectures on the multiplicity of standard graded algebras made by various authors. After introducing the conjectures, we review earlier work, especially for monomial ideals. We will briefly describe theorems of Eisenbud and Schreyer that settled these conjectures. Finally, in the last section, we will make some reductions that apply in the case of monomial ideals.

4.1 Introduction

Let $I \subseteq R = \Bbbk[V]$ be a homogeneous ideal. (Recall that V is a finite set of indeterminates.) For $0 \leq l \operatorname{pd} R/I$, let $\underline{m}_l$ and $\overline{m}_l$ denote the minimum and maximum twists of R/I at homological degree l, as in Definition 1.2.3. Let $e(R/I)$ denote the Hilbert-Samuel multiplicity of R/I; see Discussion 1.2.6. Let $c = \operatorname{ht} I$.

J. Herzog, C. Huneke and H. Srinivasan [HS98, Conjecture 2] conjectured that:

Conjecture (HHSu). Let $I \subseteq R$ be a homogeneous ideal. Let $c = \operatorname{ht} I$. Then

$$e(R/I) \leq \frac{\overline{m}_1 \overline{m}_2 \cdots \overline{m}_c}{c!}.$$

In the next section, we will discuss some of the cases where this is known. When R/I is Cohen-Macaulay, they conjectured a lower bound [HS98, Conjecture 1].

Conjecture (HHSl). Let $I \subseteq R$ be a homogeneous ideal. Let $c = \operatorname{ht} I$. If R/I is Cohen-Macaulay, then

$$e(R/I) \geq \frac{\underline{m}_1 \underline{m}_2 \cdots \underline{m}_c}{c!}.$$

Huneke and M. Miller [HM85, Theorem 1.2] proved that if R/I is Cohen-Macaulay and has a pure resolution, then the above conjectures hold, with equality. Motivated by this, J. Migliore, U. Nagel and T. Römer [MNR08, Conjecture 1.1] conjectured that:

Conjecture (HHSe). If equality holds in Conjecture (HHSu) or in Conjecture (HHSl) then R/I is Cohen-Macaulay with a pure resolution.

Now suppose further that I is a monomial ideal. Let $\mathbb{T}_\bullet$ be the Taylor resolution of R/I; see Section 1.3.2. Let τ_l be the largest twist of R appearing in T_l. Suppose $f_1, \ldots, f_m$ are the (unique) monomial minimal generators of I. Then we see that $\tau_l = \max\{\deg \operatorname{lcm}(f_{s_1}, \cdots, f_{s_l}) : 1 \leq s_1 < \cdots < s_l \leq m\}$. Herzog and Srinivasan [HS04, p. 231] conjectured that:

Conjecture (TB). Let $I \subseteq R$ be a monomial ideal with $\operatorname{ht} I = c$. Then

$$e(R/I) \leq \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

Conjecture (TB) is weaker than Conjecture (HHSu). For, using the Taylor resolution of R/I to compute $\operatorname{Tor}_l^R(R/I, \Bbbk)$, we see that $\beta_{l,j}(R/I) = 0$ for all $j > \tau_l$. Hence $\tau_l \geq \overline{m}_l$ for all $1 \leq l \leq c$.

In Section 4.5, we show that Conjecture (TB) holds for all ideals generated by quadratic monomials:

Theorem 4.1.1. *Let $I \subseteq R$ be generated by monomials of degree 2. Then*

$$e(R/I) \leq \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

Next, we show that Conjectures (HHSu) and (HHSe) hold for edge ideals of bipartite graphs:

Theorem 4.1.2. *Let $I \subseteq R$ be the edge ideal of a bipartite graph G. Then*

$$e(R/I) \leq \frac{\overline{m}_1 \overline{m}_2 \cdots \overline{m}_c}{c!}.$$

Theorem 4.1.3. *Let I be the edge ideal of a bipartite graph G. If equality holds in Conjecture (HHSu), then R/I is a complete intersection, or is Cohen-Macaulay with* $\operatorname{reg} R/I = 1$. *In either of the cases, R/I is Cohen-Macaulay and has a pure resolution.*

4.2 Earlier work

Recall that R/I is said to have a quasi-pure resolution if for each homological degree l, $\underline{m}_{l+1} \leq \overline{m}_l$; see page 23. Herzog and Srinivasan showed that if R/I is Cohen-Macaulay and has a quasi-pure resolution, then Conjectures (HHSu), (HHSl) and (HHSe) hold for R/I [HS98, Theorem 1.2]. They showed, further, that Conjectures (HHSu) and (HHSl) are true whenever $\operatorname{ht} I = 2$ and R/I is Cohen-Macaulay or when $\operatorname{ht} I = 3$ and R/I is Gorenstein. In the case of monomial ideals, Herzog and Srinivasan showed that Conjecture (HHSu) holds if I is a stable monomial ideal (in the sense of Eliahou-Kervaire [EK90]) or is a square-free strongly stable monomial ideal (in the sense of Aramova-Herzog-Hibi [AHH98]). If I belongs to any of the above classes of monomial ideals and R/I is Cohen-Macaulay, then Conjecture (HHSl) holds for I.

These conjectures have subsequently been proved in various cases. A survey appears in [FS07]. We now discuss some of the cases of monomial ideals where these conjectures have been verified. Let I be a square-free monomial ideal. I. Novik and E. Swartz [NS06] showed that if the Stanley-Reisner complex of I is a matroid complex, then Conjecture (HHSu) holds for I. See [Sta96, Section III.3] for the combinatorial definition of matroid complexes. In terms of I, we can define a matroid complex as follows. The Stanley-Reisner complex of I is a *matroid complex* if for all $\sigma \subseteq V$, the ring $\Bbbk[\sigma]/(I \cap \Bbbk[\sigma])$ is Cohen-Macaulay. They also showed that Conjecture (HHSu) holds for square-free monomial ideals I such that $\dim R/I \leq 3$.

Let Δ be a simplicial complex on the vertex set V. For $0 \leq i \leq \dim \Delta$, the *$i$-dimensional skeleton*, $\operatorname{Skel}_i(\Delta)$, of Δ is the collection of the faces of Δ of dimension at most i. Now let Δ be Stanley-Reisner complex of I. We say that Δ is *q-Cohen-Macaulay* if for all $\sigma \subseteq V$ with $|\sigma| < q$, $\Delta|_\sigma$ is Cohen-Macaulay. Let $q_i = \max\{q : \operatorname{Skel}_i(\Delta)$ is q-Cohen-Macaulay$\}$; the *Cohen-Macaulay connectivity sequence* of Δ is the sequence $(q_0, \ldots, q_{\dim \Delta})$. In the context of Conjecture (HHSl), they described the $\underline{m}_l$ of I in terms of the Cohen-Macaulay connectivity sequence of Δ. Let $d = \dim R/I$ (so that $|V| = c + d$). Then $[|V|] \setminus \{m_1, \ldots, m_c\} = \{|V| - q_0 + 1, \ldots, |V| - q_{d-1} + 1\}$[NS06, Theorem 1.3].

M. Kubitzke and V. Welker showed that Conjectures (HHSu) and (HHSl) hold for the barycentric subdivision of simplicial complexes [KW06, Theorem 1.2]. Let Δ be a simplicial complex on the vertex set V. Then the *barycentric subdivision* of Δ is the simplicial complex $\tilde{\Delta}$ on the vertex set $\{F \in \Delta : F \neq \emptyset\}$ with facets $\{\{F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_d\} : F_i \in \Delta \text{ for all } i\}$. The Stanley-Reisner ideal of $\tilde{\Delta}$ is generated by $\{FG : F, G \in \Delta, F \nsubseteq G, G \nsubseteq F\}$; hence it is a quadratic square-free monomial ideal.

4.3 Conjectures of Boij-Söderberg

Let $\mathbb{B} := \mathbb{Q}^{(|V|+1)\times\infty}$ be the vector space over $\mathbb{Q}$, spanned by a basis $e_{l,j}, 0 \leq l \leq |V|, -\infty < j < \infty$. An *(abstract) Betti table* is an element $\beta \in \mathbb{B}$; this stems from the fact that the Betti table (see p. 23 for the definition) of a finitely generated graded M belongs to $\mathbb{B}$. We say that an abstract Betti table is *pure* if it is the Betti table of a finite length module that has a pure resolution. (If M is a finitely generated R-module with $\dim M = 0$, then $\dim_{\mathbb{k}} M$ is finite; hence such modules are called finite length modules.) Herzog and Kühl [HK84, Theorem 1] showed that M is a finite length module and M has a pure resolution, with (unique) twist d_l at homological degree $l, 0 \leq l \leq |V|$, then

$$\beta_{l,d_l} = \beta_{0,d_0} \prod_{k \neq l,0} \frac{d_k - d_0}{|d_k - d_l|}. \tag{4.1}$$

M. Boij and J. Söderberg conjectured that the Betti table of a finite length module M is a non-negative rational combination of pure Betti tables [BS06, Conjecture 2.4]. This conjecture implies Conjectures (HHSu) and (HHSe) in the Cohen-Macaulay case, and Conjecture (HHSl) [BS06, Proposition 2.8]. D. Eisenbud and F.-O. Schreyer proved the above conjecture of Boij and Söderberg [ES07, Theorem 0.2]. Boij and Söderberg conjectured additionally that given any sequence $(d_0 < d_1 < \ldots < d_{|V|})$ of integers, there is a finitely length module M having a pure resolution with twists d_l at homological degree l, and graded Betti numbers given by Equation 4.1. This was proved, in characteristic zero, by Eisenbud, G. Fløystad and J. Weyman [EFW07], and, independent of the characteristic, by Eisenbud and Schreyer [ES07, Theorem 0.1].

4.4 Some Reductions for Monomial Ideals

In the next two chapters, we will give a proof of Conjecture (TB) for all quadratic monomial ideals, and of Conjecture (HHSu) for edge ideals of bipartite graphs. We will characterize the bipartite graphs whose edge ideals have a Cohen-Macaulay quotient with a quasi-pure resolution. These arguments are combinatorial.

Proposition 4.4.1. *Let $1 \le l \le c$. Then for all $x \in V$,*

a. $\overline{m}_l((I,x)) \le \overline{m}_l(I)$ *and* $\overline{m}_l((I:x)) \le \overline{m}_l(I)$.

b. $\tau_l((I,x)) \le \tau_l(I)$ *and* $\tau_l((I:x)) \le \tau_l(I)$.

Proof. Let $W = V \setminus \{x\}$ and $J = (I \cap \Bbbk[W])R$. Then $(I,x) = (J,x)$ and x is a non-zerodivisor on R/J; hence $c-1 \le \operatorname{ht} J \le c$.

(a): Let $\mathbb{G}_\bullet$ be a minimal graded free resolution of R$/J$. Denote the (graded) Koszul complex on x by $\mathbb{K}_\bullet$. Then $\mathbb{G}_\bullet \otimes \mathbb{K}_\bullet$ is a minimal graded free resolution of $R/((J,x))$; in particular, $\overline{m}_l((I,x)) = \overline{m}_l((J,x)) = \max\{\overline{m}_l(J), \overline{m}_{l-1}(J)+1\}$. Since $\operatorname{ht} J \ge c-1$, we conclude using Lemmas 1.2.4 and 1.3.8(a) that, for $1 \le l \le c-1$, $\overline{m}_l((I,x)) = \overline{m}_l(J) \le \overline{m}_l(I)$. If $\overline{m}_c(J) > \overline{m}_{c-1}(J)$, then $\overline{m}_c((I,x)) = \overline{m}_c(J) \le \overline{m}_c(I)$; otherwise, $\overline{m}_c((I,x)) = \overline{m}_{c-1}(J)+1 \le \overline{m}_{c-1}(I)+1 \le \overline{m}_c(I)$.

Lemma 1.3.8(b) implies that $\overline{m}_l((I:x)) \le \overline{m}_l(I)$.

(b): Recall that I is generated by square-free monomials $f_1, \cdots, f_m$. Let

$$(f_j : x) := \begin{cases} \frac{f_j}{x}, & \text{if } x \text{ divides } f_j, \\ f_j, & \text{otherwise} \end{cases}$$

Since $(I:x) = ((f_1:x), \cdots, (f_m:x))$ and $(I,x) = (J,x)$, the conclusions follow easily from the definition of τ_l. $\square$

Consider $I \cap R_1$, the vector space generated by the linear forms in I. Suppose that $\dim_{\Bbbk}(I \cap R_1) > 0$; then, since I is a monomial ideal, there exists $x \in V$ such that $x \in I$. Write $J = (I \cap \Bbbk[V \setminus \{x\}])R$. Then $\operatorname{ht} J = c-1$ and $I = (J,x)$. Note that $e(R/J) = e(R/I)$. From Lemma 1.3.8(a) we know that $\overline{m}_l(J) \leq \overline{m}_l(I)$ for $1 \leq l \leq c-1$. From the definition of τ_l, we see that $\tau_l(J) \leq \tau_l(I)$ for $1 \leq l \leq c-1$. Therefore it is enough to prove Conjectures (HHSl) and (TB) for J. In other words, I behaves like an ideal of height $c-1$. Hence, if $\dim_{\Bbbk}(I \cap R_1) = \delta$, we will say that I is *essentially of height $c-\delta$*.

Discussion 4.4.2. To make further reduction, we use the sequence (1.3). Let $x \in V$. If $\operatorname{ht}(I : x) > c$, then $e(R/I) = e(R/(I,x))$. In light of Proposition 4.4.1, we can replace I by I by (I,x) which is essentially of height $\leq c-1$, and prove Conjectures (HHSu) and (TB) by induction on height. We can also look at (I,x) as an ideal in $n-1$ variables. On the other hand, if $\operatorname{ht}(I,x) > c$, then $e(R/I) = e(R/(I:x))$; we then replace I by $(I:x)$ which is an ideal in $n-1$ variables. In this case, we can prove the conjectures using induction on the number of variables. Therefore, we reduce to the case that $\operatorname{ht}(I:x) = c = \operatorname{ht}(I,x)$. For later use, we record this below:

Hypothesis 4.4.3. For all $x \in V$, $\operatorname{ht}(I:x) = c = \operatorname{ht}(I,x)$; consequently, for all $x \in V$, $e(R/I) = e(R/(I,x)) + e(R/(I:x))$.

The remark about $e(R/I)$ follows from (1.3). This hypothesis is equivalent to the assumption that for all $x \in V$, there exist $\mathfrak{p}, \mathfrak{q} \in \operatorname{Unm} R/I$ such that $x \in \mathfrak{p} \setminus \mathfrak{q}$. Moreover, while proving the conjectures, we will assume, inductively, that conjectures (HHSu) and (TB) hold for $(I:x)$ and (I,x).

Proposition 4.4.4. *Let I be the edge ideal of a bipartite graph G on $V = V_1 \bigsqcup V_2$. Then Hypothesis 4.4.3 holds for I if and only if G is perfectly matched.*

Proof. If G is perfectly matched, then let $\mathfrak{p} := (x : x \in V_1)$ and $\mathfrak{q} := (x : x \in V_2)$. By Lemma 2.1.1, $\operatorname{ht}\mathfrak{p} = \mathfrak{q} = c$. For all $x \in V_1$, $(I,x) \subseteq \mathfrak{q}$ and $(I : x) \subseteq \mathfrak{q}$; the case of $x \in V_2$ is similar. Hence we see that Hypothesis 4.4.3 holds for I.

Conversely, assume that G is not perfectly matched. Since V_1 and V_2 are minimal vertex covers for G, we see that $|V_1| \geq c$ and that $|V_2| \geq c$. In light of Lemma 2.1.1, we may assume, without loss of generality, that $|V_1| > c$. In the paragraph preceding Lemma 2.1.1 we noted that there is a matching with c edges. Let $\{x_1, \cdots, x_c\} \subseteq V_1, \{y_1, \cdots, y_c\} \subseteq V_2$ be such that $x_1y_1, \cdots, x_cy_c$ is a matching of G. Pick $x \in V_1 \setminus \{x_1, \cdots, x_c\}$. Then $x_1y_1, \cdots, x_cy_c, x$ is a regular sequence in (I,x), giving $\operatorname{ht}(I,x) > c$. Hence Hypothesis 4.4.3 does not hold. $\square$

Remark 4.4.5. The proof above shows that, if I is the edge ideal of a bipartite graph such that $\operatorname{ht}(I,x) = c$ for all $x \in V$, then, $\operatorname{ht}(I : x) = c$, for all $x \in V$. This is not true for arbitrary square-free monomial ideals.

Discussion 4.4.6. For $\rho, \gamma \in \mathbb{N}$, let

$$\mu(\rho,\gamma) := \begin{cases} \frac{(2\rho+1)(2\rho+2)\cdots(\rho+\gamma)}{(\rho+1)(\rho+2)\cdots\gamma} & \rho < \gamma, \\ 1, & \text{otherwise.} \end{cases}$$

Note that $\mu(\gamma-1,\gamma) = (2\gamma-1)/\gamma < 2 = 2\mu(\gamma,\gamma)$. For any $\rho < \gamma - 1$,

$$\frac{\mu(\rho+1,\gamma)}{\mu(\rho,\gamma)} = \frac{(\rho+1)(\gamma+\rho+1)}{(2\rho+1)(2\rho+2)} > \frac{1}{2}.$$

Combining these, we conclude that

$$2\mu(\rho+1,\gamma) > \mu(\rho,\gamma), \text{ for all } \rho < \gamma \in \mathbb{N}. \tag{4.2}$$

We conclude this section with a crucial lemma that captures the main numerical argument in the proofs of Theorems 4.1.1 and 4.1.2.

Lemma 4.4.7. *Let $\rho, \gamma, \gamma_1 \in \mathbb{N}$ such that $2 \leq \rho < \gamma \leq \rho\gamma_1$ and $\rho - 1 \leq \gamma - \gamma_1$. Then*

$$2^{\rho}\mu(\rho, \gamma - 1) + 2^{\rho-1}\mu(\rho - 1, \gamma - \gamma_1) < 2^{\rho}\mu(\rho, \gamma).$$

Proof. Since $\mu(\rho, \gamma) - \mu(\rho, \gamma - 1) = \rho\mu(\rho, \gamma - 1)/\gamma$, we must show that

$$\mu(\rho - 1, \gamma - \gamma_1) < \frac{2\rho\mu(\rho, \gamma - 1)}{\gamma}. \tag{4.3}$$

We first reduce the problem to the case of $\gamma = \rho\gamma_1$ as follows. If $\gamma < \rho\gamma_1$, and if we replace γ by $\gamma + 1$, the left-hand-side and right-hand-side of (4.3) are multiplied by factors of

$$\frac{\gamma - \gamma_1 + \rho}{\gamma - \gamma_1 + 1} \quad \text{and} \quad \frac{\gamma + \rho}{\gamma + 1}$$

respectively. Both these factors are greater than 1, and the left-hand-side increases by a larger factor than the right-hand-side. Therefore, it is enough to prove the lemma when $\gamma = \rho\gamma_1$, *i.e.*, that

$$\mu(\rho - 1, \rho\gamma_1 - \gamma_1) < \frac{2\rho\mu(\rho, \rho\gamma_1 - 1)}{\rho\gamma_1}.$$

The hypothesis gives that $\gamma_1 > 1$, so we need to show that

$$\frac{(2\rho - 1)(2\rho)\cdots(\rho\gamma_1 - \gamma_1 + \rho - 1)}{\rho(\rho + 1)\cdots(\rho\gamma_1 - \gamma_1)} < \frac{2\rho}{\rho\gamma_1}\frac{(2\rho + 1)(2\rho + 2)\cdots(\rho\gamma_1 + \rho - 1)}{(\rho + 1)(\rho + 2)\cdots(\rho\gamma_1 - 1)}.$$

We can verify this by hand for $(\rho, \gamma_1) = (2,2), (2,3)$ and $(3,2)$. For all other values of ρ, γ_1, $\rho + 1 \leq \rho\gamma_1 - \gamma_1 - 1$ and we rewrite the above equation as

$$\tfrac{(2\rho-1)}{\rho} \cdot 2\rho \cdot \tfrac{(2\rho+1)}{(\rho+1)} \cdots \tfrac{(\rho\gamma_1-\gamma_1-1+\rho)}{(\rho\gamma_1-\gamma_1-1)}\tfrac{1}{(\rho\gamma_1-\gamma_1)} < \tfrac{2\rho}{\rho\gamma_1}\tfrac{(2\rho+1)}{(\rho+1)} \cdots \tfrac{(\rho\gamma_1-1+\rho)}{(\rho\gamma_1-1)}$$

which is equivalent to the following sequence of equivalent statements:

$$
\begin{aligned}
\frac{(2\rho-1)}{\rho}\frac{1}{(\rho\gamma_1-\gamma_1)} &< \frac{(\rho\gamma_1-\gamma_1+\rho)}{(\rho\gamma_1-\gamma_1)}\cdots\frac{(\rho\gamma_1-1+\rho)}{(\rho\gamma_1-1)}\frac{1}{\rho\gamma_1} \\
\frac{(2\rho-1)}{\rho} &< \frac{(\rho\gamma_1-\gamma_1+\rho)}{(\rho\gamma_1-\gamma_1+1)}\cdots\frac{(\rho\gamma_1-1+\rho)}{\rho\gamma_1} \\
\left(1+\frac{\rho-1}{\rho}\right) &< \left(1+\frac{\rho-1}{(\rho\gamma_1-\gamma_1+1)}\right)\cdots\left(1+\frac{\rho-1}{\rho\gamma_1}\right).
\end{aligned}
$$

This is indeed true, since there are γ_1 terms on the right-hand-side and each of them is at least as large as $\left(1+\frac{\rho-1}{\rho\gamma_1}\right)$. Recall that $\gamma_1>1$. □

4.5 Proof of Theorem 4.1.1

We first make some observations on how the τ_l changes with l. Let $\rho(I)$ be the length of the longest R-regular sequence in $\{f_1,\cdots,f_m\}$.

Lemma 4.5.1. *Assume Hypothesis 4.4.3. Then, for all $1\leq l\leq m$, if $\tau_l<n$, then $\tau_l>\tau_{l-1}$. For all $2\leq l\leq m-1$, we have $\tau_l-\tau_{l-1}\geq\tau_{l+1}-\tau_l$. Consequently,*

$$
\tau_l=\begin{cases} 2l, & 1\leq l\leq\rho(I) \\ \min\{\rho(I)+l,n\} & \rho(I)\leq l\leq c \end{cases}
$$

Moreover, for all $x\in V$, $\rho((I:x))<\rho(I)$.

Proof. A consequence of Hypothesis 4.4.3 is that for every $x\in V$ there is a monomial minimal generator f_j such that x divides f_j, from which the first assertion follows. To prove the second assertion, assume, by way of contradiction, and by induction on m, that m is the smallest integer m' such that there exists an ideal generated by m' quadratic monomials such that the conclusion does not hold. Write $\delta_l=\tau_l-\tau_{l-1}$; it is clear that

$0 \leq \delta_l \leq 2$. Pick l smallest such that $\delta_l < \delta_{l+1}$. If $\delta_l = 0$, then $\tau_{l+1} = \tau_l = n$. Hence $\delta_l = 1$ and $\delta_{l+1} = 2$.

We now claim that $l = m-1$. For, assume, without loss of generality, that $\tau_{l+1} = \deg \operatorname{lcm}(f_1, \cdots, f_{l+1})$. Let $J = (f_1, \cdots, f_{l+1})$. Then $\tau_l(J) \leq \tau_l(I) = \tau_{l+1}(I) - 2$. If $m > l+1$, then, by minimality of m, $\delta_2(J) = \cdots = \delta_{l+1}(J) = 2$. Hence $f_1, \cdots, f_{l+1}$ is a regular sequence, and, therefore, $\tau_j(I) = 2j, \forall j \leq l+1$ and $\delta_2(I) = \cdots = \delta_{l+1}(I) = 2$ contradicting the choice of l. Therefore $l = m-1$.

Assume that $\tau_{m-1}(I) = \deg \operatorname{lcm}(f_1, \cdots, f_{m-1})$. Let $J = (f_1, \cdots, f_{m-1})$. If $\tau_{m-2}(J) < \tau_{m-2}(I)$, then $\delta_{m-1}(J) = 2$, and hence J is generated by a regular sequence of $m-1$ quadratic monomials. Therefore $\tau_j(J) = 2j, 1 \leq j \leq m-1$. Since $\tau_j(J) \leq \tau_j(I) \leq 2j$, $\tau_j(I) = 2j, 1 \leq j \leq m-1$, contradicting the assumption that $\delta_{m-1}(I) = 1$; hence $\tau_{m-2}(J) = \tau_{m-2}(I)$. We may assume that $\tau_{m-2}(J) = \deg \operatorname{lcm}(f_2, \cdots, f_{m-1})$. Then $\tau_{m-1}((f_2, \cdots, f_m)) = 2 + \tau_{m-2}(J) > \tau_{m-1}(J) = \tau_{m-1}(I)$, leading to a contradiction.

From the above discussion, and since $\tau_1 = 2$, clearly there exists ρ such that

$$\tau_l = \begin{cases} 2l, & 1 \leq l \leq \rho \\ \min\{\rho + l, n\} & \rho \leq l \leq c \end{cases}$$

What we need to show is that ρ is the length of the longest R-regular sequence in $\{f_1, \cdots, f_m\}$. If $f_{j_1}, \cdots, f_{j_t}$ form a regular sequence, then $\tau_t = 2t$, so $\rho \geq t$. Conversely, since $\tau_\rho = 2\rho$, there exists a regular sequence of length ρ in $\{f_1, \cdots, f_m\}$.

Let $y \in V$ be such that $xy \in I$. If $f_1, \cdots, f_s$ are all the quadratic minimal generators of $(I : x)$, then none of them involves x and and y; therefore, to any regular sequence in $\{f_1, \cdots, f_s\}$, one can add xy, to get a longer regular sequence. The last statement follows immediately. $\square$

Lemma 4.5.2. *With notation as above,* $\rho(I) \geq \frac{c}{2}$.

Proof. Since $\rho(I) \geq 1$, this holds when $c = 1$. By induction on c, we may assume that for all square-free monomial ideals J with $\mathrm{ht}J < c$, $\rho(J) > \frac{\mathrm{ht}J}{2}$. Take a minimal generator xy of I. Let $J = (I \cap \Bbbk[V \setminus \{x,y\}])R$. Since xy is a non-zerodivisor on R/J, $\rho(J) = \rho(I) - 1$, and, further, since, $(J,xy) \subseteq I$, $\mathrm{ht}J < \mathrm{ht}(J,xy) \leq \mathrm{ht}I$ and Since $(I,x,y) = (J,x,y)$, $\mathrm{ht}J \geq c-2$. By induction, $\rho(J) \geq \frac{c-2}{2}$, and, therefore, $\rho(I) \geq \frac{c}{2}$. $\square$

We now prove that Conjecture (TB) holds for quadratic monomial ideals.

Theorem 4.1.1. *Let $I \subseteq R$ be generated by monomials of degree 2. Then*

$$e(R/I) \leq \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

Proof. We proceed by induction on c. If $c = 2$, the Taylor bound holds for I [HS04, Corollary 4.3], so let $c \geq 3$. As discussed in the previous section, we take I to be the edge ideal of a graph G and assume that Hypothesis 4.4.3 holds.

For all $x \in V$, notice that $e(R/(I,x))$ is the number of unmixed primes $\mathfrak{p}$ of R/I containing x. Since each such prime has height c, in the sum $\sum_{x \in V} e(R/(I,x))$, it is counted c times. Therefore

$$e(R/I) = \frac{1}{c} \sum_{x \in V} e(R/(I,x)).$$

Now suppose $\tau_c = n$. As noted earlier, (I,x) is essentially of height $\leq c-1$. Therefore, by induction and by Proposition 4.4.1(b),

$$e(R/I) \leq \frac{n}{c} \frac{\tau_1 \tau_2 \cdots \tau_{c-1}}{(c-1)!} = \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

Therefore we may further assume that $\tau_c = c + \rho(I) < n$.

We now reduce to the case that $\rho(I) < c$. If $\rho(I) = c$ then, without loss of generality, take $f_1, \cdots, f_c$ to be a regular sequence. Write $J = (f_1, \cdots, f_c)$. Since $J \subseteq I$ and $\operatorname{ht} J = c = \operatorname{ht} I$, we see that $e(R/I) \leq e(R/J) = 2^c$. From Lemma 4.5.1, $\tau_l = 2l$ for all $1 \leq l \leq c$. Hence

$$e(R/I) \leq \frac{\tau_1 \tau_2 \cdots \tau_c}{c!}.$$

In particular G is not a collection of c isolated edges, which would have given $\rho(I) = c$ and $|V| = 2c$. We pick $x \in V$ such that $\deg_G x \geq 2$. Then $(I : x)$ is essentially of height $\leq c-2$. Moreover $\rho((I : x)) < \rho(I)$, by Lemma 4.5.1. We noted earlier that (I, x) is essentially of height $\leq c-1$. Let $\rho' := \rho((I, x))$. Hence, by induction on c and by Hypothesis 4.4.3, we have

$$e(R/(I,x)) \leq \frac{2 \cdot 4 \cdots 2\rho' \cdot (2\rho'+1) \cdots (c+\rho'-1)}{(c-1)!} = 2^{\rho'} \mu(\rho', c-1),$$

which gives, after successive application of (4.2), (which is permitted since $\rho(I) < c$), $e(R/(I,x)) \leq 2^{\rho(I)} \mu(\rho(I), c-1)$. Since $\deg_G x \geq 2$ and $\rho((I : x)) \leq \rho(I) - 1$, we can conclude, by a similar argument, that $e(R/(I,x)) \leq 2^{\rho(I)-1} \mu(\rho(I)-1, c-2)$. (Notice that since $\rho(I) - 1 \leq c-2$, we can apply (4.2).)

We must show that

$$e(R/I) \leq \frac{2 \cdot 4 \cdots 2\rho(I) \cdot (2\rho(I)+1) \cdots (c+\rho(I))}{c!} = 2^{\rho(I)} \mu(\rho(I), c).$$

Since $e(R/I) = e(R/(I,x)) + e(R/(I : x))$, it suffices to show that

$$2^{\rho(I)} \mu(\rho(I), c-1) + 2^{\rho(I)-1} \mu(\rho(I)-1, c-2) < 2^{\rho(I)} \mu(\rho(I), c).$$

Set $\rho = \rho(I)$, $\gamma = c$, $\gamma_1 = 2$. Since $\frac{c}{2} \leq \rho(I) < c$, and $c \geq 3$, we see that $2 \leq \rho < \gamma \leq \rho\gamma_1$ and $\rho - 1 \leq \gamma - \gamma_1$. Applying Lemma 4.4.7 now finishes the proof. $\square$

4.6 Proof of Theorem 4.1.2

The proof of Theorem 4.1.2 involves two steps. We first reduce the problem to the case of unmixed edge ideals; in the unmixed case, we relate the multiplicity to the number of antichains in the associated directed graph.

Lemma 4.6.1. *Let G be an unmixed bipartite graph with edge ideal I. For $1 \leq l \leq r(I)$, $\overline{m}_l(I) = 2l$ and for $r(I) \leq l \leq c$, $\overline{m}_l(I) = l + r(I)$.*

Proof. Follows from Proposition 2.2.2.b, and the definition of regularity. $\square$

Proposition 4.6.2. *Let G be a Cohen-Macaulay bipartite graph with edge ideal I. Then $e(R/I) = |\mathscr{A}_{\mathfrak{d}_G}|$.*

Proof. Let $\mathfrak{p} \in \operatorname{Unm} R/I$. Let $A := \{i \in [c] : y_i \in \mathfrak{p}$ and for all $j \in [c]$ with $i \succ j, y_j \notin \mathfrak{p}\}$. Note that A is an antichain. This gives a map from $\operatorname{Unm} R/I$ to $\mathscr{A}_{\mathfrak{d}_G}$, which is injective by Lemma 2.2.4. Conversely, for any antichain A of $\mathfrak{d}_G$, the prime ideal $(x_j : j \not\succcurlyeq i$ for any $i \in A) + (y_j : j \succcurlyeq i$ for some $i \in A)$ belongs to $\operatorname{Unm} R/I$. This gives a bijection $\mathscr{A}_{\mathfrak{d}_G}$ and $\operatorname{Unm} R/I$, with the empty set corresponding to $(x_1, \cdots, x_c)$. $\square$

Discussion 4.6.3 (Closing directed graphs under transitivity). Suppose that ij and jk are edges of $\mathfrak{d}_G$; then we add an edge ik. Call the new graph $\widehat{\mathfrak{d}}$ and let $\widehat{G}$ be the bipartite graph associated to $\widehat{\mathfrak{d}}$. Let $\widehat{I}$ be the edge ideal of $\widehat{G}$. Since $I \subseteq \widehat{I}$ and $\operatorname{ht} I = \operatorname{ht} \widehat{I}$, we have that $e(R/I) \geq e(R/\widehat{I})$. In order to show that $e(R/I) = e(R/\widehat{I})$, it suffices to show that $x_i y_k \in \mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Unm} R/I$. Let $\mathfrak{p} \in \operatorname{Unm} R/I$ be such that $x_i \notin \mathfrak{p}$. Then, since $k \succ i$, by Lemma 2.2.4, $y_k \in \mathfrak{p}$, and therefore, $x_i y_k \in \mathfrak{p}$. Moreover, any coclique in $|\widehat{\mathfrak{d}}|$ is a coclique in $|\mathfrak{d}_G|$, so $\kappa(\widehat{G}) \leq \kappa(G)$. It follows from Lemma 2.2.3 and Theorem 2.2.15 that $\operatorname{reg} R/\widehat{I} = r(\widehat{I}) \leq r(I)$. We see at once from Lemma 4.6.1 that $\overline{m}_l(\widehat{I}) \leq \overline{m}_l(I), 1 \leq l \leq c$.

Lemma 4.6.4. *Let $\mathfrak{d}$ be any poset on c vertices, with order $\succ$, $\mathscr{A}$ the set of antichains in $\mathfrak{d}$ and $r = \max\{|A| : A \in \mathscr{A}\}$. Then $|\mathscr{A}| \leq 2^r \mu(r,c)$. Equality holds above, if and only if $r = 1$ or $r = c$. (See Discussion 4.4.6 for the definition of μ.)*

Proof. We prove this by induction on c. If $r = 1$, (in particular, if $c = 1$), $\mathfrak{d}$ is a chain, *i.e.*, for all $i \neq j \in [c]$, $i \succ j$ or $j \succ i$. In this case, $|\mathscr{A}| = c + 1 = 2\mu(1,c)$. If $c = r \geq 2$, then $\mathfrak{d}$ is a collection of c isolated vertices, in which every subset of $[c]$ is an antichain, *i.e.*, $|\mathscr{A}| = 2^c = 2^c \mu(c,c)$. Note that equality holds in both the cases above.

We now have $c > r \geq 2$. Pick a vertex i such that there is an antichain A with $i \in A$ and $|A| = r$. Set $\tilde{\mathfrak{d}} := \{j \in \mathfrak{d} : j \not\succ i \,\text{and}\, i \not\succ j\}$. Let $\mathfrak{d}'$ be the poset obtained by deleting i from $\mathfrak{d}$, keeping all the other elements and relations among them. Denote the respective sets of antichains by $\tilde{\mathscr{A}}$ and $\mathscr{A}'$. Now for any $A \subseteq [c]$, $A \in \mathscr{A} \setminus \mathscr{A}'$ if and only if $i \in A$ and $A \setminus \{i\} \in \tilde{\mathscr{A}}$. Therefore $\mathscr{A} = \mathscr{A}' \bigsqcup \{A \cup \{i\} : A \in \tilde{\mathscr{A}}\}$ and $|\mathscr{A}| = |\mathscr{A}'| + |\tilde{\mathscr{A}}|$.

Observe that $\max\{|A| : A \in \tilde{\mathscr{A}}\} = r - 1$. Let $r' := \max\{|A| : A \in \mathscr{A}'\}$. Then $r' \leq r$. Let $c_1 := |\mathfrak{d} \setminus \tilde{\mathfrak{d}}|$. Then $\tilde{\mathfrak{d}}$ has $c - c_1$ vertices. We note that $r - 1 \leq c - c_1$. We assume, by induction on the number of vertices, that the lemma holds for $\tilde{\mathfrak{d}}$ and $\mathfrak{d}'$, yielding

$$|\mathscr{A}| \leq 2^{r'} \mu(r', c-1) + 2^{r-1} \mu(r-1, c-c_1),$$

and, by repeated application of (4.2) from Discussion 4.4.6, (which is permitted since $r' \leq r \leq c-1$)

$$|\mathscr{A}| \leq 2^r \mu(r, c-1) + 2^{r-1} \mu(r-1, c-c_1). \tag{4.4}$$

Since $c > r \geq 2$, we must show that $|\mathscr{A}_{\mathfrak{d}}| < 2^r\mu(r,c)$; to this end, it suffices to show that

$$2^r\mu(r,c-1)+2^{r-1}\mu(r-1,c-c_1) < 2^r\mu(r,c),$$

which follows from Lemma 4.4.7 with $\rho = r, \gamma = c, \gamma_1 = c_1$. Note that by the choice of i, $c \leq rc_1$. $\square$

Theorem 4.1.2. *Let $I \subseteq R$ be the edge ideal of a bipartite graph G. Then*

$$e(R/I) \leq \frac{\overline{m}_1\overline{m}_2\cdots\overline{m}_c}{c!}.$$

Proof. In light of Discussion 4.6.3, we may assume that G is unmixed. We now replace G by its acyclic reduction $\widehat{G}$. (Discussion 2.2.6 contains the definition of acyclic reduction.) First, from Lemma 2.2.8 and Proposition 4.6.2 we see that the multiplicities remain unchanged. Let $I, \widehat{I}$ be the respective edge ideals. Note that $c = \operatorname{ht} I \geq \operatorname{ht} \widehat{I}$. Hence Remark 2.2.11 gives that $\overline{m}_l(\widehat{I}) = \overline{m}_l(I), 1 \leq l \leq \operatorname{ht} \widehat{I}$. Therefore we now assume that G is a Cohen-Macaulay bipartite graph. From Proposition 4.6.2, $e(R/I) = |\mathscr{A}_{\mathfrak{d}_G}|$. Corollary 4.6.1 gives

$$\frac{\overline{m}_1(I)\cdots\overline{m}_c(I)}{c!} = 2^{r(I)}\mu(r(I),c).$$

Since, by Theorem 2.2.15, $r(I) = \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\}$, we apply Lemma 4.6.4, with $\mathscr{A} = \mathscr{A}_{\mathfrak{d}_G}$, to finish the proof. $\square$

When can equality hold for I in the conjectured bound? The proof Theorem 4.1.2 above and Lemma 4.6.4, show that if G is a Cohen-Macaulay bipartite graph with edge ideal I, and equality holds for I, then $\operatorname{reg} R/I = c$ or $\operatorname{reg} R/I = 1$. We are now ready to prove Theorem 4.1.3. We recall that for unmixed bipartite graph G with edge ideal I, $\operatorname{reg} R/I = r(I) = \kappa(G) = \max\{|A| : A \in \mathscr{A}_{\mathfrak{d}_G}\}$ (Theorem 2.2.15).

Theorem 4.1.3. *Let I be the edge ideal of a bipartite graph G. If equality holds in Conjecture (HHSu), then R/I is a complete intersection, or is Cohen-Macaulay with* $\operatorname{reg} R/I = 1$*. In either of the cases, R/I is Cohen-Macaulay and has a pure resolution.*

Proof. We first reduce to the case that Hypothesis 4.4.3 holds. We will show that $\operatorname{ht}(I,x) = c$ for $x \in V$; this suffices, by Remark 4.4.5.

Assume, by way of contradiction, that $x \in V$ is such that $\operatorname{ht}(I,x) > c$. Then $\operatorname{ht}(I : x) = c$ and $e(R/(I : x)) = e(R/I)$. We may assume that x is not an isolated vertex of G; for otherwise, x would not have divided any minimal generator of I. Hence x has at least one neighbour, so $(I : x)$ is essentially of height at most $c-1$; see the paragraph following Proposition 4.4.1. Let $J \subseteq R$ be the ideal generated by the quadratic minimal generators of $(I : x)$. Observe that $(I : x)$ is generated by the neighbours of x, modulo J. Hence $e(R/(I : x)) = e(R/J)$. It follows from Lemma 1.3.8(a) and Proposition 4.4.1(a) that $\overline{m}_l(J) \leq \overline{m}_l((I : x)) \leq \overline{m}_l(I)$ for all $1 \leq l \leq c$. Now, $\overline{m}_l(J) > l$, for all l. Therefore equality holds for J in Conjecture (HHSu). Since $\overline{m}_l((I : x)) \geq \overline{m}_l(J)$ and $\operatorname{ht} J < c = \operatorname{ht}(I : x)$, we see that equality cannot hold for $(I : x)$, and hence, again by Proposition 4.4.1(a), for I. Therefore we may assume that Hypothesis 4.4.3 holds.

By Proposition 4.4.4, G has perfect matching. Let $\mathfrak{d}_G$ be the directed graph associated to G, as in Discussion 2.2.1. First, we construct an unmixed bipartite graph G' on the same set of vertices by closing $\mathfrak{d}_G$ under transitivity. Let I' be the edge ideal of G'. As we saw in Discussion 4.6.3, $\overline{m}_l(I') \leq \overline{m}_l(I)$. Hence equality holds for I' in Conjecture (HHSu). In particular, $r(I) = r(I')$, which, since G' is unmixed, equals $\operatorname{reg} R/I'$. Note that $\operatorname{ht} I' = c$.

Let G'' be the acyclic reduction of G'. Denote the edge ideal of G'' by I''. We know from Remark 2.2.11, Lemma 2.2.14 and Theorem 2.2.15 that $\overline{m}_l(I') = \overline{m}_l(I'')$, for all $1 \leq l \leq \operatorname{ht} I''$. If, indeed, G'' had fewer vertices than G', *i.e.*, $\operatorname{ht} I'' < c$, then equality could not have held for I' or I, for $\overline{m}_c(I') > c$. We thus see that G' is its own acyclic reduction; In other words, $\mathfrak{d}_G$ does not have any directed cycles, or, equivalently, G' is Cohen-Macaulay.

Since equality holds for G', we see from Lemma 4.6.4 that $r(I') = c$ or $r(I') = 1$. If $r(I') = c = \operatorname{ht} I'$, then $\mathfrak{d}_{G'}$, and, hence, $\mathfrak{d}_G$, are an antichains. Therefore $I = I' = (x_1y_1, \ldots, x_cy_c)$; see Remark 2.2.16. Since all the minimal generators of I have the same degree, R/I has a pure resolution.

If $r(I') = 1$, then $\mathfrak{d}_{G'}$ and, hence, $\mathfrak{d}_G$ have precisely one source vertex and one sink vertex. With that, $1 = r(I') = \kappa(G') \leq \kappa(G) \leq r(I) = 1$ if and only if $\mathfrak{d}_G$ is a chain. In other words, R/I is Cohen-Macaulay with $\operatorname{reg} R/I = 1$, which, evidently, has a pure resolution. $\square$

Chapter 5

Monomial Support and Projective Dimension

In this chapter, we will look at the question of bounding the projection dimension of homogeneous ideals in a polynomial ring from numerical data about the ideal. We will introduce the notion of a monomial support for an ideal, and construct an example that shows that any bound for projective dimension based on the size of a monomial support for an ideal is at least exponential. This work was done jointly with G. Caviglia, and appears in [CK08].

Various questions about bounding invariants of projective resolutions from numerical data about ideals have been raised by different researchers. *E.g.*, M. Stillman had asked whether there is a bound on projective dimension of a homogeneous ideal, if only the degrees of the minimal generators are known. Observe that these questions seek for bounds independent of the number of variables; if the number of variables is known, then it is an upper bound for projective dimension, by the Hilbert syzygy theorem.

In the next section, we will define monomial supports, and introduce the question. In Section 5.2, we present the example showing that the bound is at least exponential. Finally, in Section 5.3, we consider some variations on this question.

5.1 Monomial Supports

Let $I \subset R = \Bbbk[V]$ be a homogeneous ideal, minimally generated by homogeneous polynomials $f_1, \ldots, f_m$. A *monomial support* of I is the set of monomials appearing with non-zero coefficients in at least one of the $f_1, \ldots, f_m$. An ideal may have different monomial supports. For example, consider $R = \Bbbk[x,y,z]$ and $I = (xz + xy + y^2, x^3 + y^3)$. Then $\{xz, y^2, x^3, y^3\}$ is a monomial support of I. We can, however, rewrite $I = (xz + y^2, x^3 - xyz - xy^2)$, from which we get the monomial support $\{xz, y^2, x^3, xyz, xy^2\}$.

C. Huneke asked the following. Assume that for some choice of minimal generators of I, it has a monomial support consisting of N monomials. Then is $\operatorname{pd} R/I \leq N$? Here again, as in Stillman's question, the number of variables is unspecified. His motivation for this question was the case of monomial ideals. Suppose that I is a monomial ideal, generated by N monomials, then $\operatorname{pd} R/I \leq N$. This follows from the Taylor resolution of R/I which has length N (see Section 1.3.2). We will answer this question in the negative, showing that any bound is at least exponential (Theorem 5.2.3).

While counting monomials in the support of a set of minimal homogeneous generators of I, we choose to count them with multiplicity. *I.e.*, we count each monomial with multiplicity equal to the number of minimal generators in which it appears. Our decision of taking the multiplicity into account while counting the monomials in the support of I is only a matter of exposition. For example, let $g_1, \ldots, g_N$ be N distinct monomials, all of the same degree, and let $f_1, \ldots, f_m$, be $\Bbbk$-linear combinations of $g_1, \ldots, g_N$, such that $f_1, \ldots, f_m$ are linearly independent over $\Bbbk$. Then $f_1, \ldots, f_m$ is a set of minimal generators of the ideal $(f_1, \ldots, f_m)$. Choose any monomial order on R. By doing an elimination with respect to this monomial order, analogous to the one used in computing a reduced Gröbner basis, we can rewrite $f_1, \ldots, f_m$ such that initial monomial of f_i does not belong to the monomial support of f_j whenever $j \neq i$. In this way, we get a monomial support for $(f_1,\dots,f_m)$ of at most $m(N-m+1)$ monomials, counted with multiplicity. The maximum value of this quantity, as a function of m, is $\lfloor (\frac{N+1}{2})^2 \rfloor$, which occurs when $m = \lfloor (N+1)/2 \rfloor$. In general, if we have N distinct monomials in a monomial support of an homogeneous ideal I, then we would have at most $\lfloor (\frac{N+1}{2})^2 \rfloor$ of them when counted with multiplicity; this is because the above function is quadratic and the worst possible case happens precisely when the ideal is generated by forms having the same degree. What we show in the next section is that projective dimension of a homogeneous ideal can grow at least exponentially with the size of any monomial support; hence, counting the monomials with multiplicity does not change the nature of the bound.

5.2 Main Example

Let $d \geq 2$ and let $n_i \geq 2$, $1 \leq i \leq d$ be positive integers. Denote by $\mathscr{I}$ the index set $\{1,\dots,n_1\} \times \cdots \times \{1,\dots,n_d\}$. We take V to be the d-dimensional array of variables $\{x_\nu : \nu \in \mathscr{I}\}$. Let $R = \Bbbk[V]$. Let

$$s_{ij} := \prod_{\substack{\nu \in \mathscr{I} \\ \nu_i = j}} x_\nu,\ 1 \leq j \leq n_i,\ 1 \leq i \leq d.$$

We will call s_{ij} *the jth slice in the ith direction*. Figure 5.1 illustrates the above definitions for a $3 \times 4 \times 2$ array. (We will define $\ell_{(3,1)}$ in the figure later).

Let $I = (s_{i1} - s_{ij} : 2 \leq j \leq n_i, 1 \leq i \leq d-1) + (s_{dj} : 1 \leq j \leq n_d)$. Then:

Proposition 5.2.1. *With notation as above,* $\operatorname{depth} R/I = 0$.

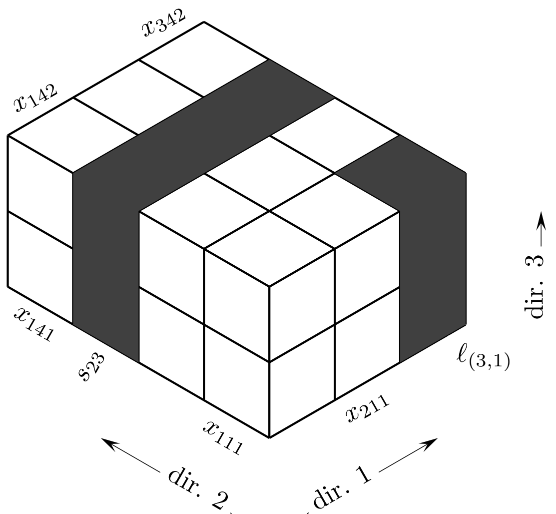


Figure 5.1: Slices of a $3 \times 4 \times 2$ array

Proof. Write $\mathfrak{m}$ for the homogeneous maximal ideal of R and let

$$s := \prod_{i=1}^{d-1} \prod_{j=2}^{n_i} s_{ij}.$$

Note that s is the product of the variables not appearing in the first slices in each of the directions $1, \ldots, d-1$. We claim that $s \in (I : \mathfrak{m}) \setminus I$. Indeed, if $(I : \mathfrak{m}) \neq I$, then $\mathfrak{m}$ is an associated prime of R/I, so $\operatorname{depth} R/I = 0$.

We first reduce the proof to the case when $\operatorname{char} \mathbb{k} = 0$, as follows. Since I is generated by monomials and binomials with ± 1 as coefficients, a Gröbner basis for I, and hence the ideal membership problem $s \in (I : \mathfrak{m}) \setminus I$ are independent of the characteristic of the field. See [Eis95] for the definition of a Gröbner basis and the ideal membership problem. We assume, from now on, that $\operatorname{char} \mathbb{k} = 0$.

Let $\nu \in \mathscr{I}$. Using the binomial relations in I, we can write

$$s \equiv \prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i} \mod I$$

where $\widehat{\cdot}$ denotes omitting the variable from the product. Consider the slice $s_{d\nu_d} = \prod_{\substack{\mu \in \mathscr{I} \\ \mu_d = \nu_d}} x_\mu$. If $\mu \neq \nu \in \mathscr{I}$ is such that $\mu_d = \nu_d$, then there exists $1 \leq i \leq d-1$ such that $\mu_i \neq \nu_i$ which gives $x_\mu | (s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i})$. Hence $s_{d\nu_d} | ((\prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i}) x_\nu)$, so, $((\prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i}) x_\nu) \in I$. This implies that $s \in (I \colon \mathfrak{m})$.

We now show that $s \notin I$. Let A be the tableau

$$\begin{array}{lcl} a_{11} & \cdots & a_{1n_1} \\ a_{21} & \cdots & a_{2n_2} \\ & \ddots & \\ a_{(d-1)1} & \cdots & a_{(d-1)n_{(d-1)}} \end{array}$$

of non-negative integers. We use *tableau* loosely here; we only mean that the rows of A possibly have different number of elements. For such a tableau A, we say it *satisfies row condition* $(c_1, \ldots, c_t)$ if the sum of the elements on the ith row is $c_i - 1$.

Let $\mathscr{P} := \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_{d-1}\}$. For each $p \in \mathscr{P}$, we define a monomial

$$\ell_p := \prod_{\substack{\nu \in \mathscr{I} \\ \nu_i = p_i, 1 \leq i \leq d-1}} x_\nu$$

(see Figure 5.1 for an illustration of $\ell_{(3,1)}$ in the $3 \times 4 \times 2$ case). Further, write $|p|_A$ for $\sum_{i=1}^{d-1} a_{ip_i}$.

Let

$$F = \sum_{A:A \text{ satisfies } (n_1, \ldots, n_{d-1})} \left\{ \prod_{p \in \mathscr{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}. \tag{5.1}$$

We let R act on itself by partial differentiation with respect to the variables. We show below that, under this action, $s \notin (0 :_R F)$ while $I \subseteq (0 :_R F)$ from which we conclude that $s \notin I$, thus proving the proposition.

For any tableau A that satisfies the row condition $(n_1, \ldots, n_{d-1})$, write τ_A for the corresponding monomial term that appears in F (see (5.1)). Let

$$A_s := \begin{matrix} 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ & & \ddots & \\ 0 & 1 & \cdots & 1 \end{matrix}$$

Then $s = \alpha \tau_{A_s}$ for some non-zero rational number α If $A \neq A_s$, then s contains a variable that τ_A does not contain, so $s \circ F = s \circ \tau_{A_s} = 1$. Hence $s \notin (0 :_R F)$.

For any $1 \leq j \leq n_d$, $s_{dj} \circ F = 0$. For, any A that appears in the summation of (5.1) has at least one $p_A \in \mathscr{P}$ such that $|p_A|_A = 0$. Hence the variables in ℓ_{p_A} do not appear in τ_A. However, s_{dj} contains one such variable, so, $s_{dj} \circ \tau_A = 0$. Hence $s_{dj} \circ F = 0$.

Observe that any slice $s_{ij}, 1 \leq i \leq d-1, 1 \leq j \leq n_i$ can be written as a product of $\ell_p, p \in \mathscr{P}$ as follows:

$$s_{ij} = \prod_{\substack{\nu \in \mathscr{I} \\ \nu_i = j}} x_\nu = \prod_{\substack{1 \leq \nu_{i'} \leq n_{i'} \\ 1 \leq i' \leq d-1 \\ \nu_i = j}} \underbrace{\left[\prod_{1 \leq \nu_d \leq n_d} x_\nu \right]}_{\ell_{(\nu_1, \ldots, \nu_{d-1})}} = \prod_{\substack{p \in \mathscr{P} \\ p_i = j}} \ell_p.$$

Let $\mathscr{P}_{ij} = \{p \in \mathscr{P} : p_i = j\}$. Then $s_{ij} \circ F = \left(\prod_{p \in \mathscr{P}_{ij}} \ell_p \right) \circ F$. Therefore to differentiate with respect to s_{ij}, we may differentiate with respect to all $\ell_p, p \in \mathscr{P}$, sequentially.

Let $1 \leq i \leq d-1, 1 \leq j \leq n_i$ and $q \in \mathscr{P}_{ij}$. Then

$$\begin{aligned}
\ell_q \circ F &= \ell_q \circ \sum_{A:A \text{ satisfies } (n_1,\ldots,n_{d-1})} \left\{ \prod_{p \in \mathscr{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \\
&= \sum_{A:A \text{ satisfies } (n_1,\ldots,n_{d-1})} \left\{ \frac{(|q|_A)^{n_d}}{(|q|_A!)^{n_d}} \ell_q^{(|q|_A-1)} \prod_{\substack{p \in \mathscr{P} \\ p \neq q}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}.
\end{aligned}$$

Therefore,

$$s_{ij} \circ F = \sum_{A:A \text{ satisfies } (n_1,\ldots,n_{d-1})} \left\{ \prod_{p \in \mathscr{P}_{ij}} \frac{(|p|_A)^{n_d}}{(|p|_A!)^{n_d}} \ell_p^{(|p|_A-1)} \prod_{p \notin \mathscr{P}_{ij}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}. \tag{5.2}$$

We can write $\{A : A \text{ satisfies } (n_1, \ldots, n_{d-1})\} = \{A : a_{ij} = 0\} \bigcup \{A : a_{ij} \neq 0\}$. Every row of A contains at least one zero. If $a_{ij} = 0$, then there is a $p \in \mathscr{P}_{ij}$ such that $|p|_A = 0$. Therefore there is no contribution from those A with $a_{ij} = 0$ in the RHS of (5.2). Moreover, if $a_{ij} \neq 0$ then $|p|_A \neq 0$. Hence

$$s_{ij} \circ F = \sum_{\substack{A \text{ satisfies } (n_1,\ldots,n_{d-1}) \\ a_{ij} \neq 0}} \left\{ \prod_{p \in \mathscr{P}_{ij}} \frac{1}{[(|p|_A - 1)!]^{n_d}} \ell_p^{(|p|_A-1)} \prod_{p \notin \mathscr{P}_{ij}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}.$$

There is a 1-1 correspondence between $\{A : A \text{ satisfies } (n_1, \ldots, n_{d-1}), a_{ij} \neq 0\}$ and $\{A : A \text{ satisfies } (n_1, \ldots, n_i - 1, \ldots, n_{d-1})\}$. Using this we can write

$$s_{ij} \circ F = \sum_{A \text{ satisfies } (n_1,\ldots,n_i-1,\ldots,n_{d-1})} \left\{ \prod_{p \in \mathscr{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}. \tag{5.3}$$

Note that this representation of $s_{ij} \circ F$ is independent of j; hence $(s_{i1} - s_{ij}) \circ F = 0$ for all $1 \leq i \leq d-1$ and $2 \leq j \leq n_i$. Therefore $I \subseteq (0 :_R F)$. $\square$

It now follows from the Auslander-Buchsbaum formula (Proposition 1.2.5) that

Corollary 5.2.2. *With notation as above,* $\operatorname{pd} R/I = n_1 \cdots n_d$. □

Parenthetically, we note that the ideal we construct has $n_i - 1$ generators of degree $n_1 \cdots \widehat{n_i} \cdots n_d$, for $1 \le i \le d-1$ and n_d generators of degree $n_1 \cdots n_{d-1}$.

Consider the case when $n_1 = \cdots = n_d = n$. Then the ideal is generated by $(n-1)(d-1)$ binomials and n monomials, and, hence, has a monomial support of $2(n-1)(d-1)+n$.

Theorem 5.2.3. *Any upper bound for projective dimension of an ideal supported on N monomials counted with multiplicity is at least* $2^{N/2}$.

Proof. Given a positive integer N, choose $n = 2$ variables in each of $d = \frac{N}{2}$ dimensions, and construct R and I as above. Then $\operatorname{pd} R/I = 2^{N/2}$. □

5.3 Further Questions

Motivated by the example in the previous section, we raise the following question:

Question 5.3.1. Suppose $I \subseteq R$ has a monomial support of N monomials, counted with multiplicity. Then what is a good upper bound for $\operatorname{pd} R/I$?

However, we can ask a similar question, by appealing to Gröbner bases and initial ideals. Fix a monomial order $>$ on R. Since for all i and j, $\beta_{i,j}(R/\operatorname{in}_> I) \ge \beta_{i,j}(R/I)$ (see Theorem 1.3.1), we see that $\operatorname{in}_> I$ has at least as many generators as I has, and that $\operatorname{pd}(R/\operatorname{in}_> I) \ge \operatorname{pd}(R/I)$. From the Taylor resolution of $R/\operatorname{in}_> I$, we can see that $\operatorname{pd}(R/\operatorname{in}_> I) \le \sum_j b_{1,j}(R/\operatorname{in}_> I)$, which is the number of minimal generators of $\operatorname{in}_> I$. Hence we can pose the following question.

Question 5.3.2. Does there exist a function $\zeta_n : \mathbb{N} \to \mathbb{N}$ such for all homogeneous ideals I in n variables having a support of N monomials, I has at most $\zeta_n(N)$ forms in a Gröbner basis in some monomial order $>$? If such a ζ_n exists, how does it vary with n?

If it were true that, for a fixed N, $\sup\{\zeta_n(N) : n \in \mathbb{N}\}$ exists, then we can use that as an upper bound for the projective dimension of ideals with a support of N monomials.

Chapter 6

Alexander Duality and Serre's Property

This chapter is devoted to giving an alternate proof of a theorem of K. Yanagawa, showing that for a square-free monomial ideal I in a polynomial ring R, R/I has Serre's property (S_i) if and only its Alexander dual $I^\star$ has linear syzygies up to homological degree i. The work here was done independently of and without the knowledge of Yanagawa's work. This had, in turn, generalized an earlier result of J. Eagon and V. Reiner that R/I is Cohen-Macaulay if and only if $I^\star$ has linear resolution.

6.1 Introduction

For a finitely generated R-module M, we say that M *satisfies Serre's property* (S_i) if for all $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \dim M_\mathfrak{p}\}$. We adopt the convention that the zero module has property (S_i) for all i.

Remark 6.1.1. Our definition of property (S_i) follows [EGA, IV, 5.7.2] and [BH93, Section 2.1]. There is another definition of Serre's condition (S_i), used in [EG85, Section 0.B]: a module M is satisfies Serre's condition (S_i) if $\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \dim R_\mathfrak{p}\}$, for all $\mathfrak{p} \in \operatorname{Spec} R$.

For any homogeneous ideal $I \subseteq R$, we say that I *satisfies property* $(N_{c,i})$ (after [EK89, p. 158]) if all the minimal generators of I have degree c and a minimal graded free resolution of I is linear up to homological degree $i-1$. This definition is independent of the choice of the resolution, because I satisfies property $(N_{c,i})$ if and only if $\operatorname{Tor}_l^R(\Bbbk, I)_j = 0$ for all $0 \leq l \leq i-1$ and for all $j \neq l+c$. K. Yanagawa proved the following theorem:

Theorem 6.1.2 ([Yan00b, Corollary 3.7])**.** *Let $I \subseteq R$ be a square-free monomial ideal with* $\operatorname{ht} I = c$. *Then for $i > 1$, the following are equivalent:*

a. R/I satisfies property (S_i).

b. The Alexander dual $I^\star$ satisfies $(N_{c,i})$.

Yanagawa proved the above result by relating these properties through local and Matlis duality. Our proof uses the (S_i)-locus of R/I.

Remark 6.1.3. N. Terai [Ter99] (see Proposition 1.3.6) gave a generalization of the Eagon-Reiner theorem; we require this in our proof of Theorem 6.1.2. For two other results generalizing the Eagon-Reiner theorem, see Herzog-Hibi [HH99, Theorem 2.1(a)] and Herzog-Hibi-Zheng [HHZ04, Theorem 1.2(c)].

Remark 6.1.4. We can extend the statement to include the case $i = 1$ by replacing the statement (a) by "R/I satisfies property (S_i) and I is unmixed" (*i.e.*, for all the associated primes $\mathfrak{p}$ of R/I, $\dim R/\mathfrak{p}$ is independent of $\mathfrak{p}$). Since R/I is reduced, it always satisfies property (S_1). Hence if I is unmixed, then $I^\star$ is generated by monomials of degree c; this is property $(N_{c,1})$ for $I^\star$. For larger i, the hypothesis that I is unmixed becomes superfluous: for any ideal I, not necessarily homogeneous, if R/I satisfies property (S_2), then I is unmixed [EGA, IV, 5.10.9].

For a commutative ring A, we say that $\operatorname{Spec} A$ is *connected in codimension k*, if for all ideals $\mathfrak{a} \subseteq A$ with $\operatorname{ht} \mathfrak{a} > k$, $\operatorname{Spec} A \setminus \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{a} \subseteq \mathfrak{p}\}$ is connected, and that A is *locally connected in codimension k* if $A_\mathfrak{p}$ is connected in codimension k for all $\mathfrak{p} \in \operatorname{Spec} A$.

It is known [Har62, Corollary 2.4] that for any ideal I, not necessarily homogeneous, if R/I satisfies property (S_2), then $\operatorname{Spec} R/I$ is locally connected in codimension 1. For square-free monomial ideals, we prove the converse, giving the following equivalence:

Theorem 6.1.5. *Let $R = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\operatorname{Spec} R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2).*

6.2 Free resolutions and the locus of non-(S_i) points

Many results in this section are part of folklore. We take R to be an arbitrary regular domain, and M a finitely generated R-module with a finite free resolution

$$\mathbb{F}_\bullet : \qquad 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \; .$$

Let $c = \operatorname{codim} M$. For $1 \leq l \leq p$, set $r_l := \sum_{j=l}^{p} (-1)^{j-l} \operatorname{rk} F_j$ and $I_l := \sqrt{I_{r_l}(\phi_l)}$, where, for a map ϕ of free modules of finite rank, and a natural number t, $I_t(\phi)$ is ideal generated by the $t \times t$ minors of ϕ and $\sqrt{\ }$ denotes taking the radical of an ideal.

Remark 6.2.1. Since R is a domain, M has a well-defined rank. We apply [BE73, Lemma 1] to conclude that M is projective if and only if $I_1 = R$. We see immediately that the exact sequence $\left(0 \longrightarrow \operatorname{Im} \phi_l \longrightarrow F_{l-1} \longrightarrow \operatorname{coker} \phi_l \longrightarrow 0\right) \otimes_R R_\mathfrak{p}$ splits — we say that $\phi_l \otimes_R R_\mathfrak{p}$ *splits* if this happens — if and only if $I_l \nsubseteq \mathfrak{p}$. If $\phi_l \otimes_R R_\mathfrak{p}$ splits, then so does every $\phi_{l'} \otimes_R R_\mathfrak{p}$ for $l' \geq l$. Hence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p$. Additionally, if R is local, with maximal ideal $\mathfrak{m}$, and M is not free, then $\operatorname{pd} M = \max\{l : 1 \leq l \leq p \text{ and } I_l \subseteq \mathfrak{m}\}$.

First we determine the Cohen-Macaulay locus of M, which is an open subset of $\operatorname{Spec} R$; see [EGA, IV, 6.11.3]. Let

$$J_{CM}(M) := \bigcap_{k=c+1}^{p} \left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right), \tag{6.1}$$

taking $J_{CM}(M) = R$ if the intersection is empty.

Proposition 6.2.2. *For all $\mathfrak{p} \in \operatorname{Spec} R$, $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $J_{CM}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $l = \operatorname{codim} M_\mathfrak{p} + 1$. First, $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k < l$; otherwise, we would get an ideal $\mathfrak{q} \subseteq \mathfrak{p}$ with $\mathfrak{q} \in \min M$ and $\operatorname{ht} \mathfrak{q} < \operatorname{codim} M_\mathfrak{p}$, which is a contradiction. We now see that $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{codim} M_\mathfrak{p}$, or, equivalently (by Remark 6.2.1), $I_l \not\subseteq \mathfrak{p}$, or, equivalently (by Remark 6.2.1, again), $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k \geq l$, or, equivalently (by above), $J_{CM}(M) \not\subseteq \mathfrak{p}$. □

In order to determine the (S_i)-locus of M, we first define $\Lambda_i = \Lambda_i(M)$ to be the set of all $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{q}$ is minimal over $I_l + J_{CM}(M)$ for some $l > \operatorname{ht} \mathfrak{q} - i$. Note that Λ_i is finite. Now let $J_{(S_i)}(M) = \bigcap_{\mathfrak{q} \in \Lambda_i} \mathfrak{q}$, taking $J_{(S_i)}(M) = R$ if $\Lambda_i = \emptyset$.

Proposition 6.2.3. *For all $\mathfrak{p} \in \operatorname{Spec} R$, $\mathfrak{p} \in \mathscr{U}_{(S_i)}(M)$ if and only if $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $\Lambda_i \cap \mathfrak{p} := \{\mathfrak{q} \in \Lambda_i : \mathfrak{q} \subseteq \mathfrak{p}\}$. Since $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$, we need to show that $\mathfrak{p} \in \mathscr{U}_{(S_i)}(M)$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$.

Let $\mathfrak{q} \in \Lambda_i \cap \mathfrak{p}$. Let $l > \operatorname{ht} \mathfrak{q} - i$ be such that $\mathfrak{q}$ is minimal over $I_l + J_{CM}(M)$. We apply Remark 6.2.1 to the regular local ring $(R_\mathfrak{q}, \mathfrak{q}R_\mathfrak{q})$ to conclude that $\operatorname{pd}_{R_\mathfrak{q}} M_\mathfrak{q} > \dim R_\mathfrak{q} - i$, and, by the Auslander-Buchsbaum formula, that $\operatorname{depth} R_\mathfrak{q} < i$. Since $J_{CM}(M) \subseteq \mathfrak{q}$, $M_\mathfrak{q}$ is not Cohen-Macaulay. Hence $M_\mathfrak{q}$ does not have property (S_i), so $\mathfrak{p} \notin \mathscr{U}_{(S_i)}(M)$.

Conversely, if $\mathfrak{p} \notin \mathscr{U}_{(S_i)}(M)$, then there exists $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\operatorname{depth} M_{\mathfrak{q}} < \min\{i, \dim M_{\mathfrak{q}}\}$. Then $M_{\mathfrak{q}}$ is not Cohen-Macaulay, *i.e.*, $J_{CM}(M) \subseteq \mathfrak{q}$, and $\operatorname{pd}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} > \dim R_{\mathfrak{q}} - i$. By Remark 6.2.1, there exists $l > \operatorname{ht} \mathfrak{q} - i$ such that $I_l \subseteq \mathfrak{q}$. Let $\mathfrak{q}'$ be minimal such that $I_l + J_{CM}(M) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$. Since $\mathfrak{q}'$ is minimal over $I_l + J_{CM}(M)$ and $l > \operatorname{ht} \mathfrak{q}' - i$, we see that $\mathfrak{q}' \in \Lambda_i \cap \mathfrak{p}$. $\square$

Remark 6.2.4. Suppose that $\operatorname{ht} \mathfrak{p} = c$ for all $\mathfrak{p} \in \min M$, *i.e.*, that $\operatorname{Ann} M$ is unmixed. Then $J_{CM}(M) = I_{c+1} + \sqrt{\operatorname{Ann} M}$. If $M = R/I$ for some radical ideal I, then $r_1 = 1$ and $I_1 = I$, so we get $J_{CM}(R/I) = I_{c+1}$. Hence Λ_i consists of those primes $\mathfrak{q}$ minimal over I_l for some $l \geq c+1$ with $\operatorname{ht} \mathfrak{q} < l - i$.

Discussion 6.2.5. Let $R = \mathbb{k}[x_1, \cdots, x_n]$, taken with standard grading, and M a finitely generated graded R-module. Let $\mathbb{F}_\bullet$ be a *graded* free resolution of M, with maps of degree 0. Then the $I_{r_l}(\phi_l)$ are homogeneous: to show this, it is enough to show that if F and G are graded free modules of same finite rank and $\phi : F \to G$ is a map of degree 0, then $\det \phi$ is homogeneous. Indeed, giving bases $f_1, \cdots, f_r$ for F and $g_1, \cdots, g_r$ for G, we can write $\phi = \begin{bmatrix} a_{ij} \end{bmatrix}$. If $a_{ij} \neq 0$, then $\deg a_{ij} = \deg g_j - \deg f_i$. Since $\det \phi = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ (where, S_r is a permutation group of r elements, and $\operatorname{sgn}(\sigma)$ is the sign of a permutation σ), it suffices to show that $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ is independent of σ, whenever $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$. This is true, since if $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$, then $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)} = \sum_{i=1}^{r} \left(\deg g_{\sigma(i)} - \deg f_i\right) = \sum_{i=1}^{r} (\deg g_i - \deg f_i)$, which is independent of σ. Radicals of homogeneous ideals are homogeneous. Minimal prime ideals of M are homogeneous. Therefore the ideals $J_{CM}(M)$ and $J_{(S_i)}(M)$ are homogeneous. Minimal prime ideals of homogeneous ideals are homogeneous, so the Cohen-Macaulay and (S_i)-loci of M are determined by homogeneous prime ideals. Hence to determine whether M has property (S_i), (or, is Cohen-Macaulay), it suffices to check this at homogeneous prime ideals. We remark here that the above argument carries over *mutatis mutandis* to the situation of multigrading, for instance, when $M = R/I$ for a monomial ideal I.

6.3 Proofs of Theorems

An immediate corollary to Hochster's formula (see Discussion 1.3.7) is that $\operatorname{depth} R/J = 1$ if and only if Δ is not connected: indeed, the Auslander-Buchsbaum formula implies that $\operatorname{depth} R/J = 1$ if and only if $\operatorname{Tor}^R_{n-1}(\Bbbk, R/J) \neq 0$. Since $\operatorname{Tor}^R_i(\Bbbk, R/J)_\sigma = 0$ if $|\sigma| \leq i$, Hochster's formula gives the equivalence with $\operatorname{Tor}^R_{n-1}(\Bbbk, R/J)_{\{x_1,\cdots,x_n\}} \neq 0$, and, again, with $\widetilde{\mathrm{H}}_0(\Delta;\Bbbk) \neq 0$, which is equivalent to Δ being disconnected.

Lemma 6.3.1. *With notation as above,*

a. For all $1 \leq l \leq n$, $(I : x_l)^\star = (I^\star \cap \Bbbk[x_1, \cdots, \widehat{x_l}, \cdots, x_n])R$.

b. If R/I satisfies (S_i), then, for all $1 \leq l \leq n$, $R/(I : x_l)$ satisfies (S_i).

Proof. (a): Associated primes of $(I : x_l)$ are exactly those of I not containing x_l. Hence while computing the dual, we take the generators not involving x_l.

(b): It suffices to show that $J_{(S_i)}(R/(I : x_l)) = R$. By way of contradiction, if $J_{(S_i)}(R/(I : x_l)) \neq R$, then let $\mathfrak{p}$ be a minimal prime ideal over $J_{(S_i)}(R/(I : x_l))$; hence $(R/(I : x_l))_\mathfrak{p}$ does not have property (S_i). Since no monomial minimal generator of $(I : x_l)$ is divisible by x_l, $\mathfrak{p}$ is a monomial ideal not containing x_l; see Discussion 6.2.5. Therefore $(R/(I : x_l))_\mathfrak{p} \simeq (R/I)_\mathfrak{p}$, which has property (S_i), a contradiction. □

We are now ready to prove Theorem 6.1.2.

Theorem 6.1.2. *Let $I \subseteq R$ be a square-free monomial ideal with $\operatorname{ht} I = c$. Then for $i > 1$, the following are equivalent:*

a. R/I satisfies property (S_i).

b. The Alexander dual $I^\star$ satisfies $(N_{c,i})$.

Proof. We prove both the directions by induction on n. Let $n=3$. For any non-zero ideal $I \subseteq R = \Bbbk[x_1,x_2,x_3]$, if R/I satisfies (S_2) (equivalently, since $\dim R/I \le 2$, (S_i) for all $i \ge 2$), then R/I is Cohen-Macaulay, and, hence $\operatorname{pd} R/I = \operatorname{ht} I$. By Proposition 1.3.6, we see that $\operatorname{reg} I^\star = \operatorname{ht} I$; however, since $I^\star$ is generated by monomials of degree $\operatorname{ht} I$, $I^\star$ has a linear resolution; in particular, $I^\star$ has property $(N_{c,2})$. Conversely, if $I^\star$ has property $(N_{c,2})$, and $c=1$, then R/I is a complete intersection, and Cohen-Macaulay. If $c=2$, then $\dim R/I = 1$. One-dimensional reduced Noetherian local rings are Cohen-Macaulay.

(a) $\implies$ (b): By way of contradiction, assume that $I^\star$ does not have the property $(N_{c,i})$. By induction, assume that n is the least integer for which there is such a counter-example. By Lemma 6.3.1(a), $(I:x_l)^\star$ satisfies $(N_{c,i})$ for all $1 \le l \le n$. Now, since I does not have $(N_{c,i})$, there is a (square-free) multidegree σ and $j \le i-1$ such that $|\sigma| > j+c$ and $\beta_{j,\sigma}(I^\star) \neq 0$. We now claim that $\sigma = \{x_1,\cdots,x_n\}$: for, if, say, $x_1 \notin \sigma$, then let Δ be the Stanley-Reisner complex of $I^\star$, and $\widetilde{\Delta}$ of $(I^\star \cap \Bbbk[x_2,\cdots,x_n])R$. Then, by applying Hochster's formula, we have

$$
\begin{aligned}
\beta_{j,\sigma}(I^\star) &= \dim_\Bbbk \widetilde{\mathrm{H}}_{|\sigma|-j-2}(\Delta|_\sigma;\Bbbk) = \dim_\Bbbk \widetilde{\mathrm{H}}_{|\sigma|-j-2}(\widetilde{\Delta}|_\sigma;\Bbbk) \\
&= \beta_{j,\sigma}((I^\star \cap \Bbbk[x_2,\cdots,x_n])R) \\
&= \beta_{j,\sigma}((I:x_1)^\star)
\end{aligned}
$$

contradicting the fact that $(I:x_1)^\star$ satisfies $(N_{c,i})$. Hence $\sigma = \{x_1,\cdots,x_n\}$, and, therefore, $j < n-c = \dim R/I$. By choice, $j < i$. Moreover, $\operatorname{reg} I^\star \ge n-j$. By Proposition 1.3.6, $\operatorname{pd} R/I \ge n-j$, and, therefore $\operatorname{depth} R/I \le j$, contradicting the hypothesis that R/I satisfies (S_i).

(b) $\implies$ (a) : By way of contradiction, assume that R/I does not satisfy (S_i). We may again assume that n is the least number of variables where such a counter-example exists. Since $I^\star$ satisfies $(N_{c,i})$, $(I:x_l)^\star$ has $(N_{c,i})$ for all $1 \leq l \leq n$. By choice of n, $R/(I:x_l)$ satisfies (S_i) for all $1 \leq l \leq n$.

Now let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\operatorname{depth}(R/I)_\mathfrak{p} < \min\{i, \dim(R/I)_\mathfrak{p}\}$. If $x_l \notin \mathfrak{p}$, then, $(R/I)_\mathfrak{p} \simeq (R/(I:x_l))_\mathfrak{p}$. Hence $\operatorname{depth}(R/I)_\mathfrak{p} \geq \min\{i, \dim(R/I)_\mathfrak{p}\}$. Therefore $\mathfrak{p} = \mathfrak{m}$. Hence $\operatorname{depth} R/I < \min\{i, \dim R/I\}$. By Auslander-Buchsbaum formula, $\operatorname{pd} R/I > n-i$. Again, by the result of Terai, $\operatorname{reg} I^\star > n-i$, *i.e.*, there exists j and a multidegree σ such that $\beta_{j,\sigma}(I^\star) \neq 0$ and $|\sigma| - j > n-i$. By Hochster's theorem, non-zero Betti numbers are in square-free multidegrees, so, $|\sigma| \leq n$. Hence $j < i$, contradicting the hypothesis that $I^\star$ has $(N_{c,i})$. □

Before we proceed, we observe that if $\dim R/I \geq 2$ and R/I is connected in codimension 1, then Stanley-Reisner complex Δ of I is connected; in fact, it is *strongly connected*, *i.e.*, for any two faces F and F' of Δ of maximal dimension, we can find a sequence $F_0 = F, F_1, \cdots, F_r = F'$ of faces of maximal dimension such that for all $1 \leq i \leq n-1$, $F_i \cap F_{i-1}$ is a face of codimension 1 in F_i and F_{i-1}. To prove this, it suffices, using the correspondence between faces of Δ and prime ideals containing I [MS05, Theorem 1.7], to show that for any $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Ass} R/I$, there is a sequence $\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{p}_1, \cdots, \mathfrak{p}_r = \mathfrak{p}'$ of associated primes of R/I such that for all $1 \leq i \leq n-1$, $\operatorname{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) = \operatorname{ht}\mathfrak{p}_i + 1 = \operatorname{ht}\mathfrak{p}_{i+1} + 1$. This follows from setting $d = 2$ in [EGA, IV, 5.10.8]. Finally, since R/I is connected in codimension 1, it is equidimensional; this is the content of the proof of [EGA, IV, 5.10.9]. Hence every vertex of Δ is in some face of maximal dimension, so Δ is connected.

Theorem 6.1.5. *Let $R = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then* $\operatorname{Spec} R/I$ *is locally connected in codimension* 1 *if and only if R/I satisfies property (S_2).*

Proof. We will show that if $\operatorname{Spec} R/I$ is locally connected in codimension 1, then R/I has property (S_i); the other implication is already known [Har62, Corollary 2.4]. If $c \geq n-1$, then it is clear that R/I is locally connected in codimension 1 and that R/I has property (S_2). Therefore we will assume that $c \leq n-2$.

We proceed by induction on n. Let $n = 3$. It is easy to verify that any unmixed monomial ideal in three variables in locally connected in codimension 1. Since $c = 1$, R/I is a complete intersection and, hence has property (S_2). Now assume that $n > 3$.

We first observe that for all $1 \leq l \leq n$, $\operatorname{Spec} R/(I : x_l)$ is locally connected in codimension 1, because $\operatorname{Spec} R/(I : x_l)$ is homeomorphic to $\operatorname{Spec}(R/I)_{x_l}$, which is locally connected in codimension 1, $(R/I)_{x_l}$ being a localization of R/I. Since x_l does not divide any minimal generator of $(I : x_l)$, we note that $(I : x_l)$ is extended from the subring $\Bbbk[x_1, \cdots, \widehat{x_l}, \cdots, x_n] \subseteq R$. By induction $R/(I : x_l)$ has property (S_2). Now let $\mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \neq \mathfrak{m}$. We can then pick $x_l \notin \mathfrak{p}$. Since $(R/I)_{\mathfrak{p}} \simeq (R/(I : x_l))_{\mathfrak{p}}$, we see that $\operatorname{depth}(R/I)_{\mathfrak{p}} \geq \min\{2, \dim(R/I)_{\mathfrak{p}}\}$. It remains to show that $\operatorname{depth} R/I \geq 2$, *i.e.*, that the Stanley-Reisner complex Δ of I is connected, which follows from the preceding discussion. $\square$

6.4 Discussion

First, there is a generalization of Alexander duality for arbitrary monomial ideals, introduced by E. Miller; see [MS05, Chapter 5]. It is worth determining whether an analogue of Theorem 6.1.2 for all monomial ideals is true.

Secondly, in the context of Theorem 6.1.5, we note the following result of M. Kalkbrener and B. Sturmfels [KS95, Theorem 1]: for all prime ideals $\mathfrak{p} \subseteq R$ and monomial orders $>$, $R/\sqrt{\operatorname{in}_> \mathfrak{p}}$ is connected in codimension 1. Initial ideals were defined in Section 1.3.1. Kalkbrener and Sturmfels use the language of simplicial complexes, and say that the Stanley-Reisner complex of $\sqrt{\operatorname{in}_> \mathfrak{p}}$ is *strongly connected*; to see that these notions are equivalent, see [Hun07, Appendix 1] by A. Taylor. One may wonder whether the stronger property of locally connected in codimension 1 holds for radicals of initial ideals. The following example shows that this is not true.

Example 6.4.1 (C. Huneke). Let $R = \Bbbk[a,b,c,d]$ and $I = (d^2, c^2 - ad, b^2 - bc, ac - bc, ab, a^2 + ad - bd + cd)$. Set $S = R[T_1, \ldots, T_6]$, and $\mathfrak{p} \subseteq S$ to be the kernel of the map $S \twoheadrightarrow R[It]$, sending $T_1 \mapsto d^2$, $T_2 \mapsto c^2 - ad$, $T_3 \mapsto b^2 - bc$, $T_4 \mapsto ac - bc$, $T_5 \mapsto ab$ and $T_6 \mapsto a^2 + ad - bd + cd$). In other words, $\mathfrak{p}$ defines the Rees algebra $R[It]$. Let $>$ be the lex order on S with $T_1 > T_2 > \ldots > T_6 > d > c > b > a$. Then

$$\begin{aligned}\sqrt{\operatorname{in}_> \mathfrak{p}} = (&T_1T_3, T_1T_5, T_2T_3T_5, T_3d, T_1c, T_2T_5c, T_2dc, T_4dc, T_5dc,\\ &T_1b, T_2b, T_3cb, T_1a, T_3a, T_2da, T_2ca, T_4ba).\end{aligned}$$

Since $\operatorname{ht}\mathfrak{p} = 5$ and $\sqrt{\operatorname{in}_> \mathfrak{p}}$ is unmixed (by Kalkbrener-Sturmfels), $\left(\sqrt{\operatorname{in}_> \mathfrak{p}}\right)^\star$ is generated by square-free monomials of degree 5. We can however see, from computing a free resolution in [M2], that the Alexander dual $\left(\sqrt{\operatorname{in}_> \mathfrak{p}}\right)^\star$ does not have property $(N_{5,2})$, so $R/\sqrt{\operatorname{in}_> \mathfrak{p}}$ does not have (S_2).

One of the problems that we can identify from the above example is to determine prime ideals $\mathfrak{p}$ such that $R/\sqrt{\operatorname{in}_> \mathfrak{p}}$ has property (S_2) for all monomial orders $>$. It is also unknown whether there are monomial orders for which $R/\sqrt{\operatorname{in}_> \mathfrak{p}}$ has property (S_2) for all prime ideals $\mathfrak{p}$.

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List of Notation

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