Infinite games on finite graphs

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3 July 2006

Reactive systems

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  Compute output as a function of inputs

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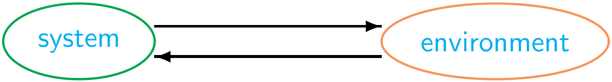

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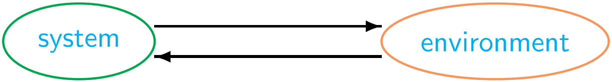
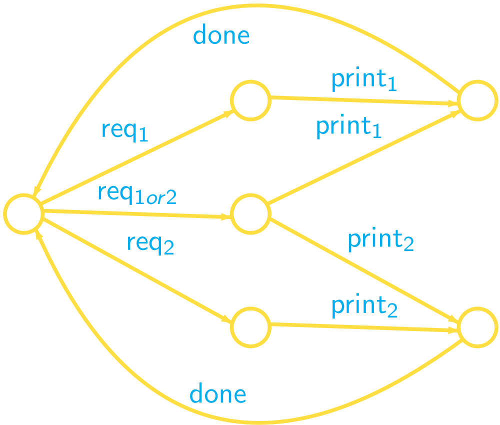


- Describe continuous interaction between system and environment as an infinite game

Modelling reactive systems

A scheduler that allocates requests to two printers



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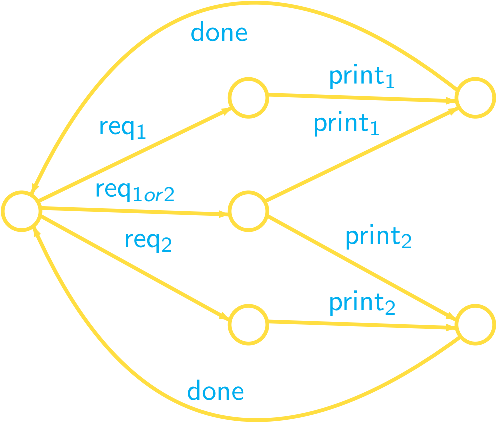

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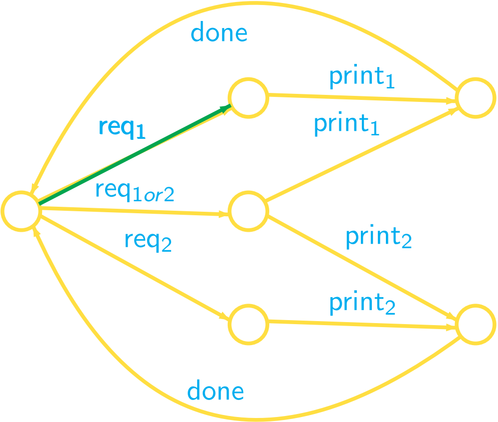

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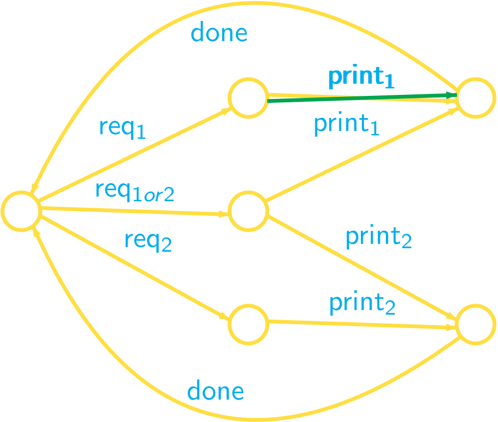

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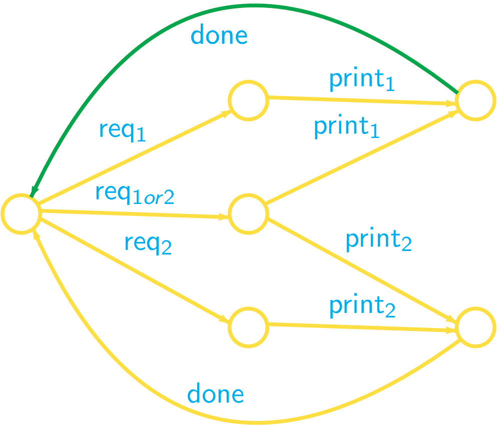

Computation is a sequence of actions

req_1 print_1 done

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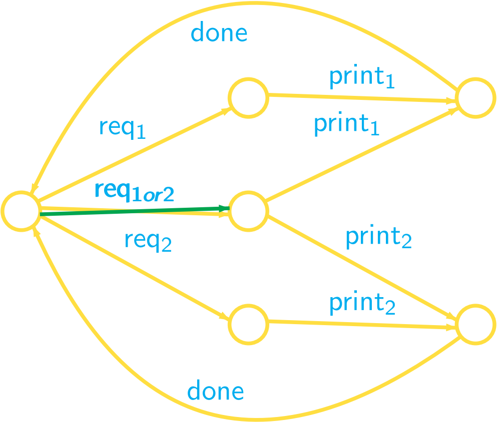

Computation is a sequence of actions

req$_1$ print$_1$ done req$_{1or2}$

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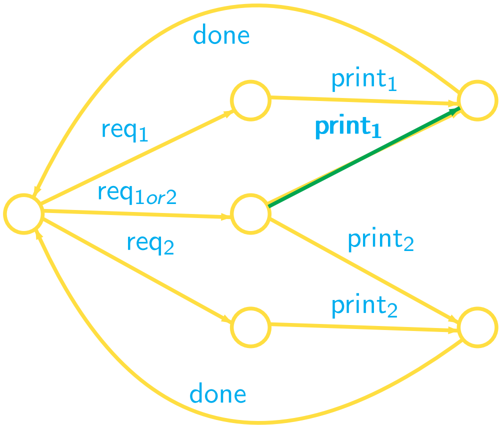

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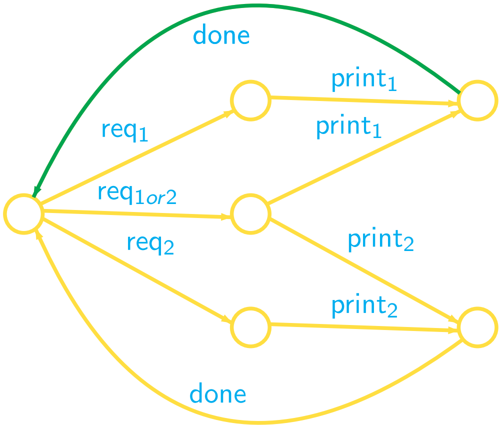

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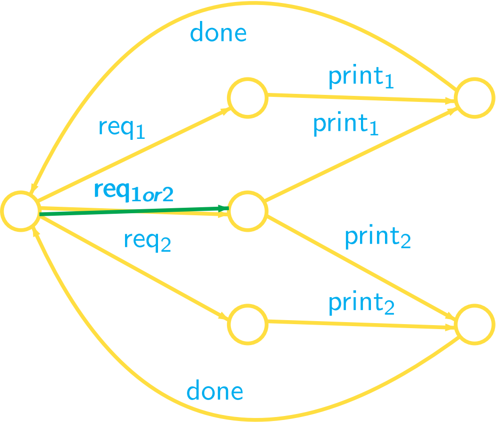

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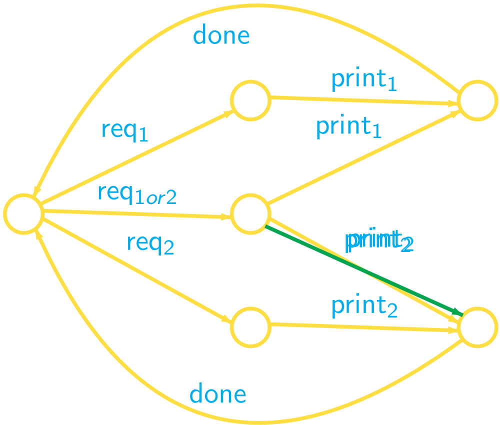

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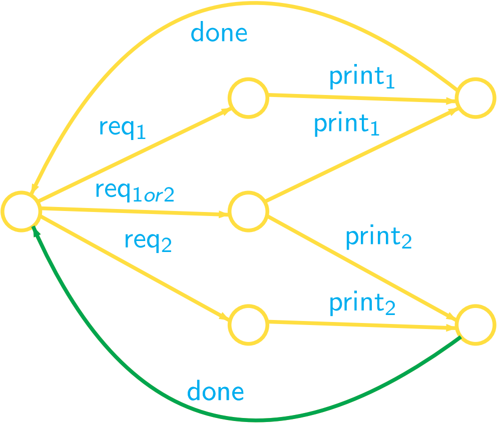

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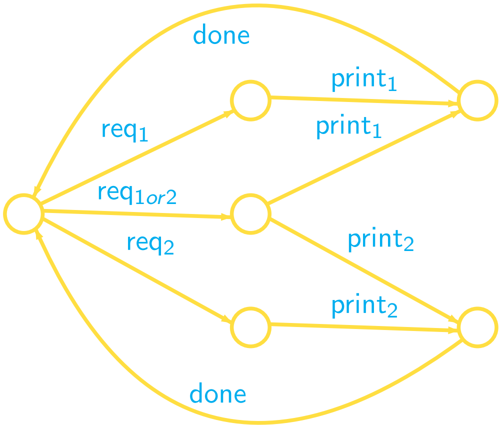
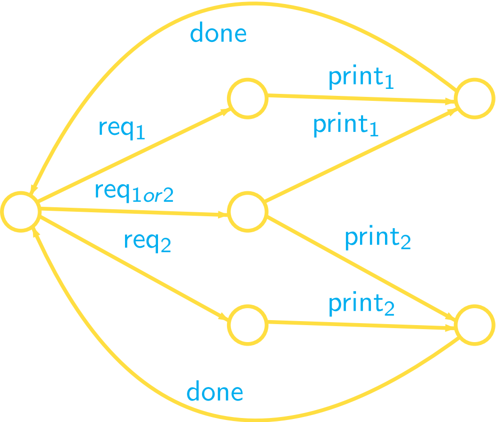


Computation is a sequence of actions , typically infinite

req$_1$ print$_1$ done req$_{1or2}$ print$_1$ done req$_{1or2}$ print$_2$ done ...

Desirable and undesirable computations



Printer 1 is colour printer, Printer 2 is black and white

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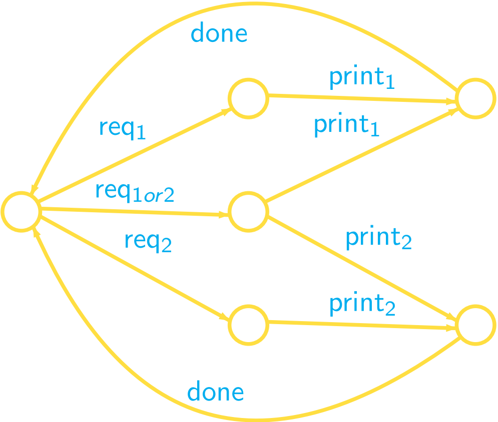

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Schedule jobs to minimize cost

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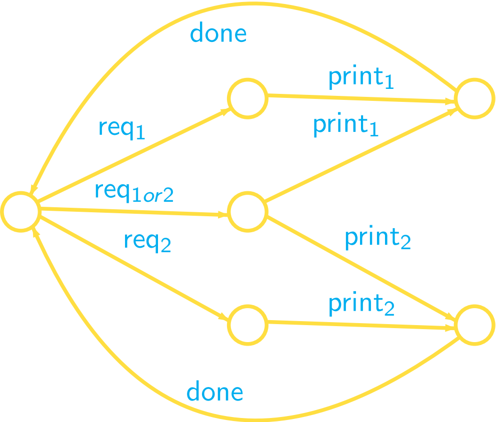

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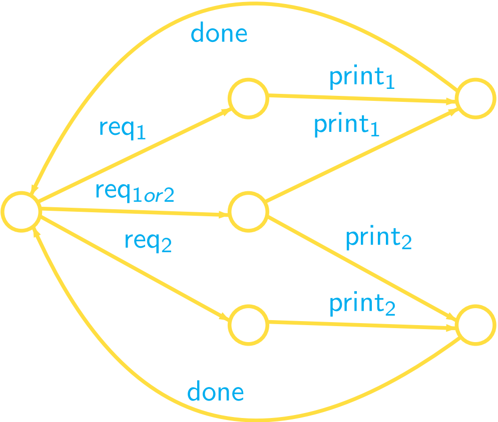

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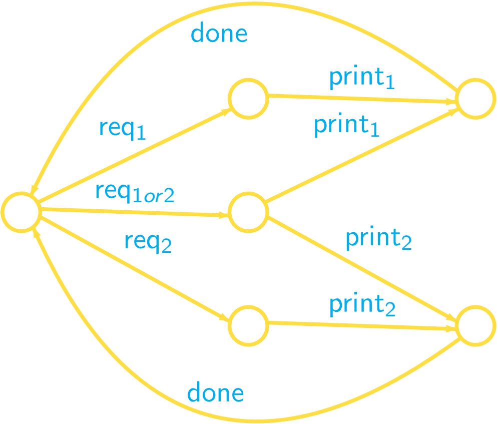

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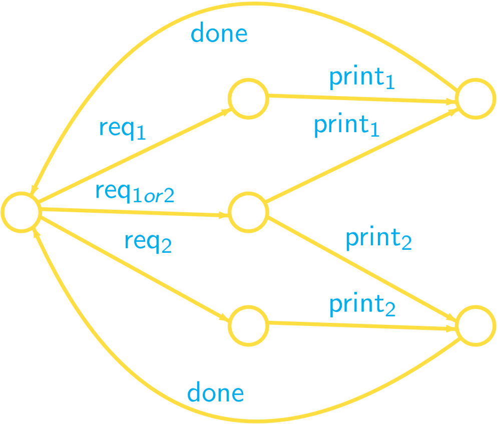
Controllable and uncontrollable actions



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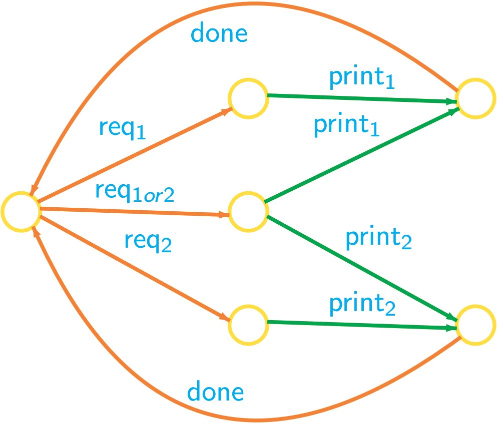

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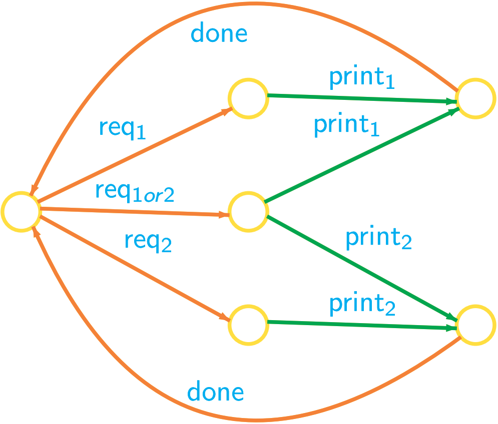

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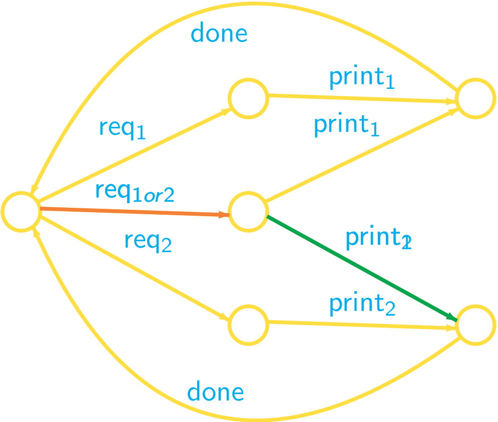

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- Given a system and an objective, is there a strategy to select controllable actions such that the objective is realized?

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  - Controllability is a winning strategy for system

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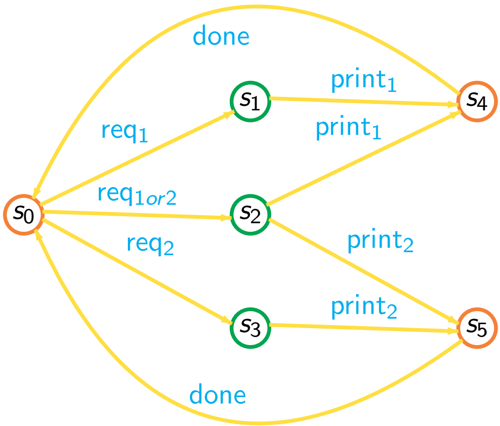
  - Otherwise Player 1 wins

Infinite games on finite graphs . . .



- Player 0 plays at green positions, Player 1 at orange positions

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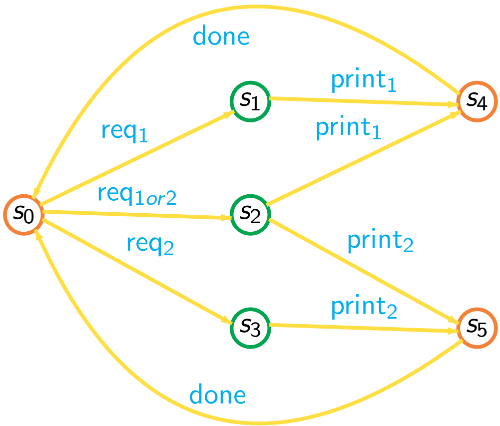

- Player 0 plays at green positions, Player 1 at orange positions
- Winning condition: every s_2 is immediately followed by s_5

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- This is $\mathbf{Reach(G)}$

Winning conditions — recurrence (Büchi condition)

- Want to visit a set **G** of good states infinitely often

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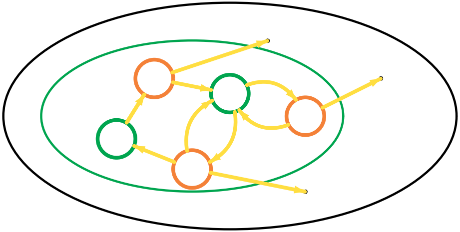

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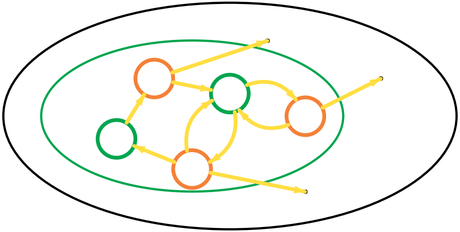

- Player 0 cannot leave the trap and Player 1 can force Player 0 to stay in the trap

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- This is a special case of a very general result for infinite games [Martin, 1975]

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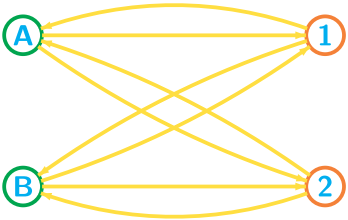

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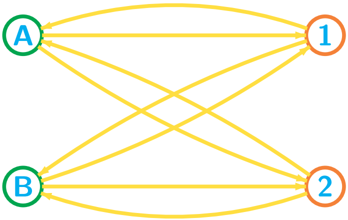

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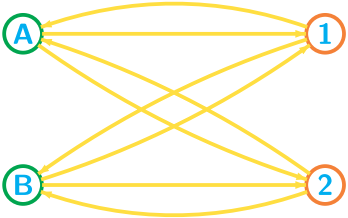

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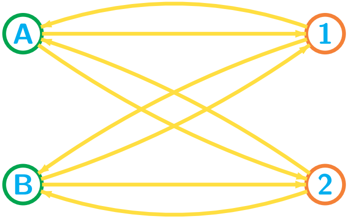

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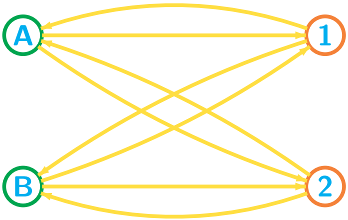

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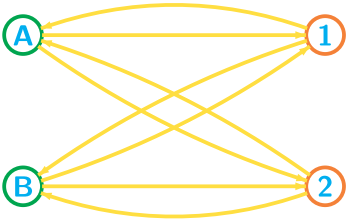

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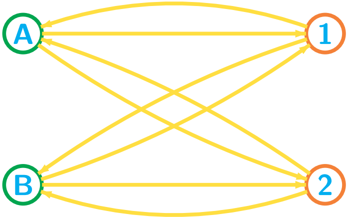

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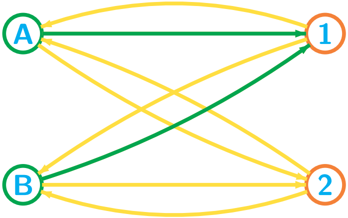

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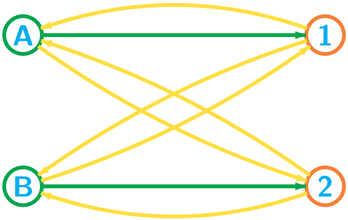

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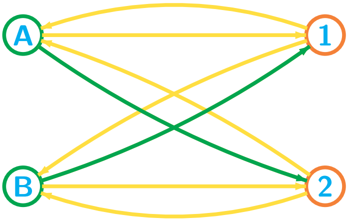

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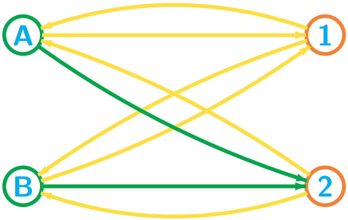

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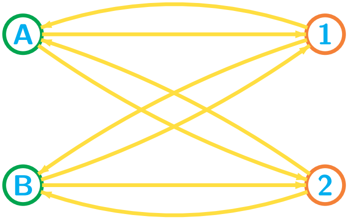

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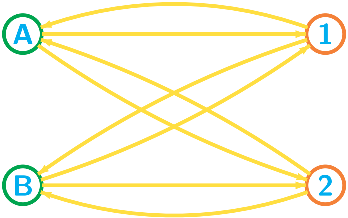

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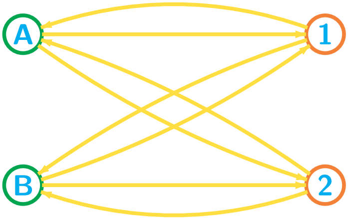

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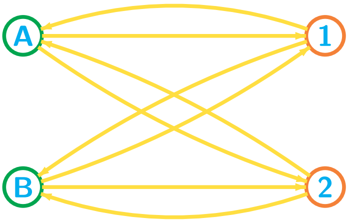

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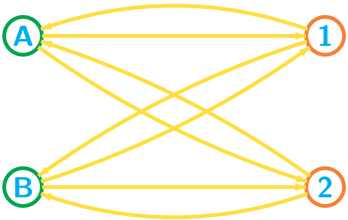

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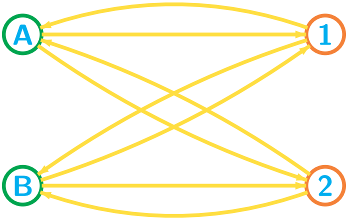

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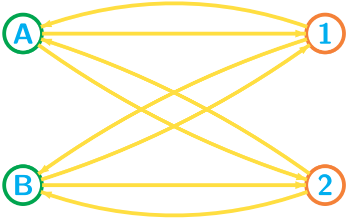

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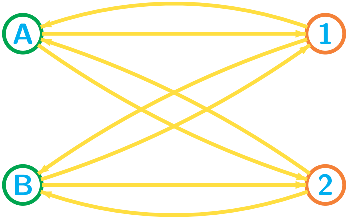

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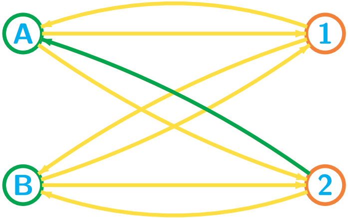

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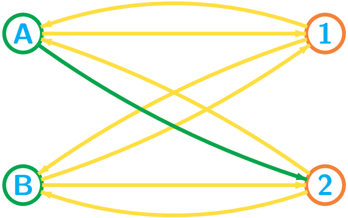

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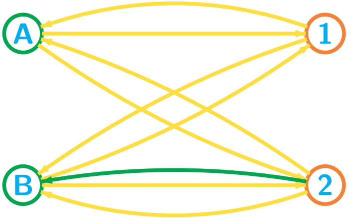

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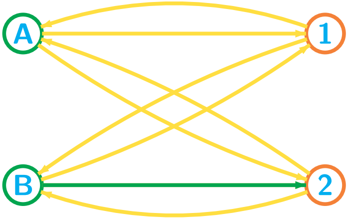

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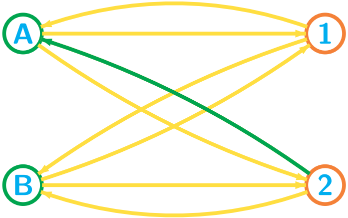

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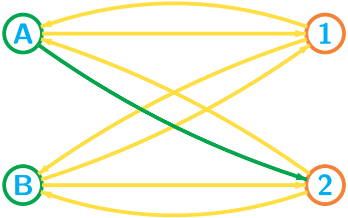

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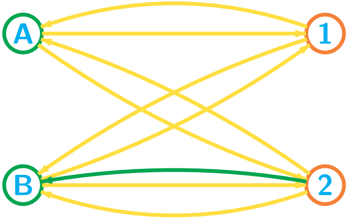

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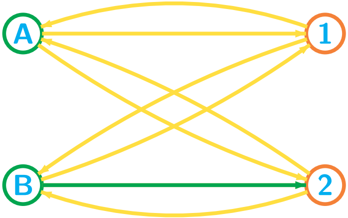

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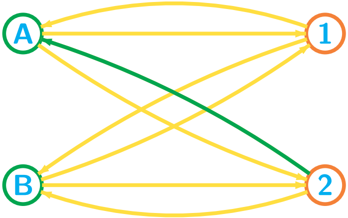

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- Muller condition $(G_1, G_2, \ldots, G_k)$

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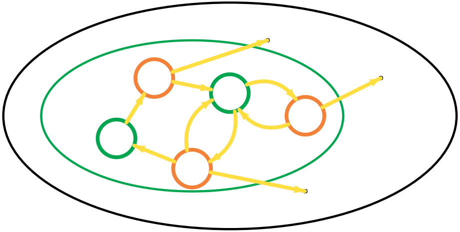
Parity games have memoryless winning strategies

- Trap for Player 0 : set of states **X** such that
  - For Player 0, all moves from **X** lead back to **X**
  - For Player 1, at least one move from **X** leads back to **X**
  - Player 0 cannot leave the trap and Player 1 can force Player 0 to stay in the trap



- Trap for Player 1 : symmetric
- For any **X**, **S** $\setminus$ **Reach(X)** is a 0 trap

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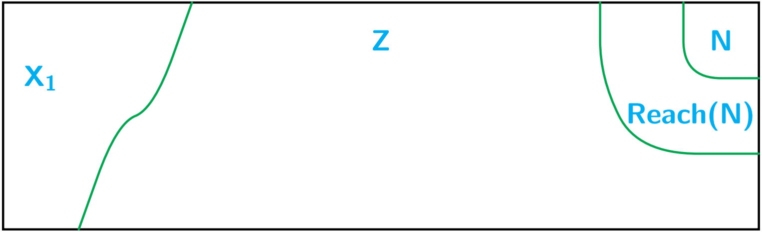

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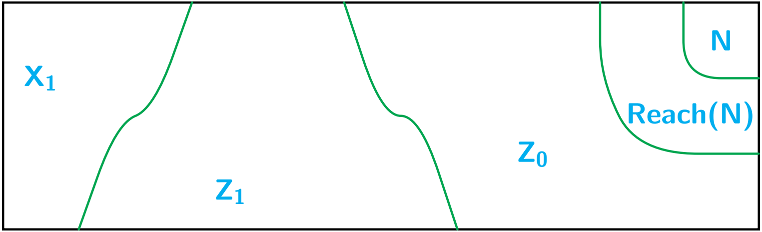

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- Z is a subgame with parities $< \mathbf{n}$
  Inductively, split Z as 1 paradise Z_1 and 0 paradise Z_0

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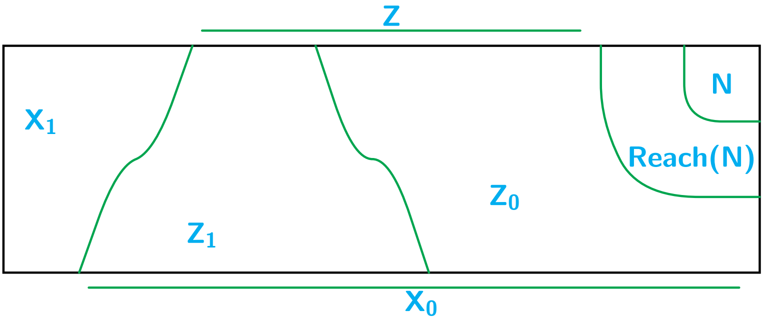

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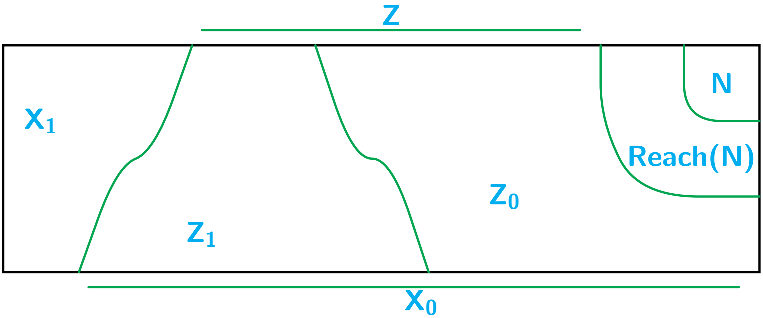

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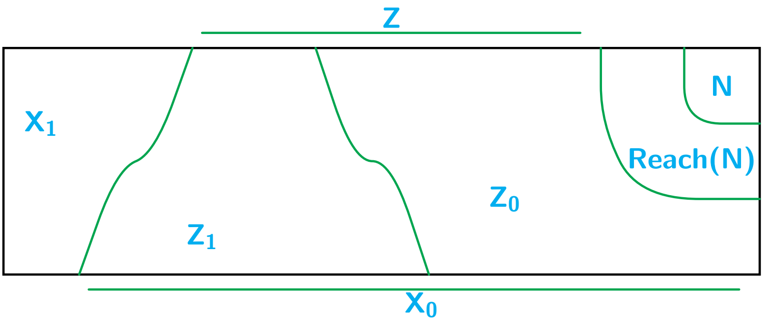

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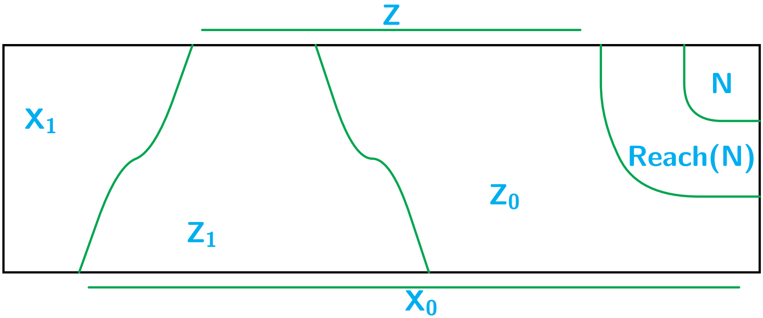

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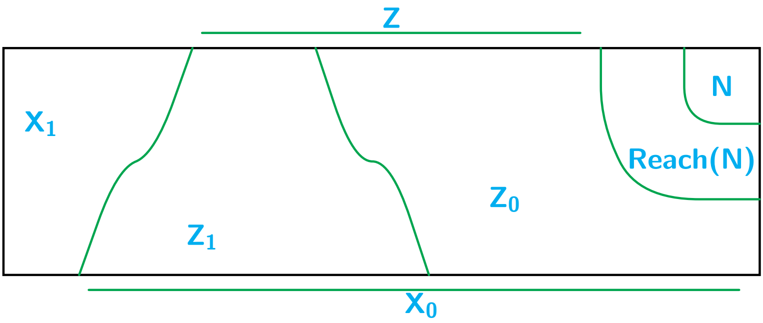

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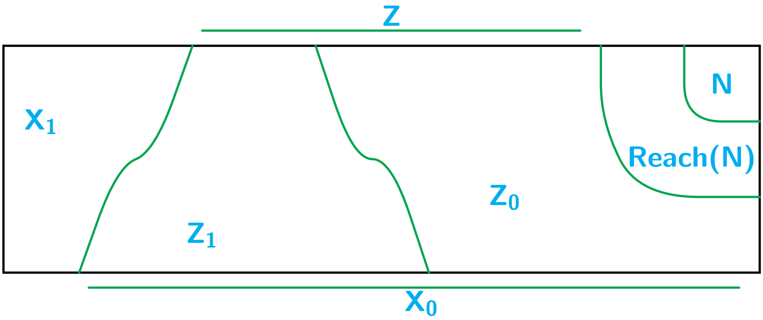

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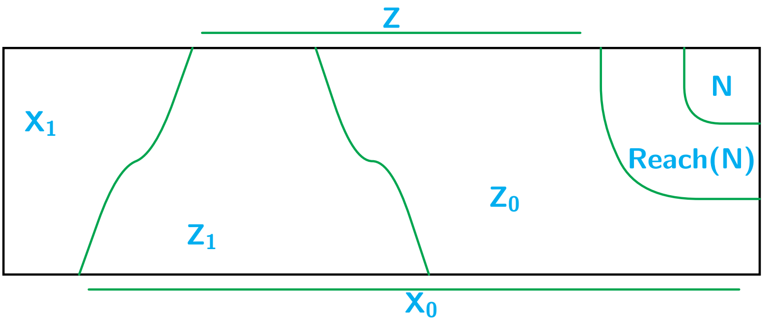

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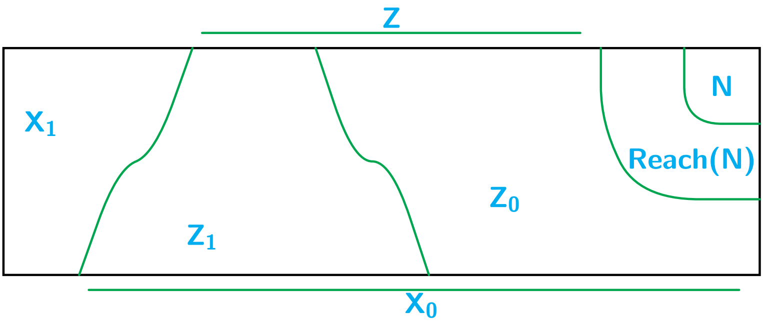

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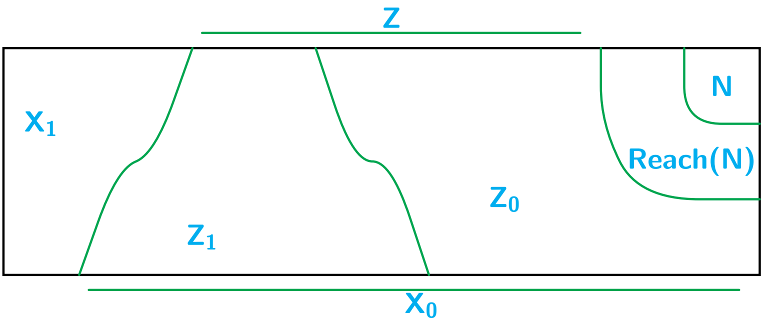

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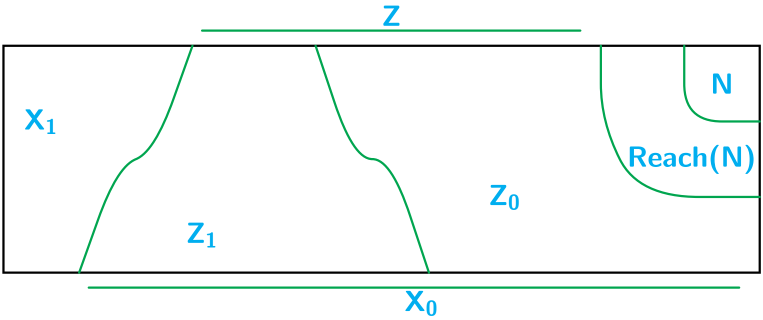

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- Recursively partition positions into 0 and 1 paradise, starting with X_1 empty

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- What is the complexity of constructing a memoryless winning strategy for parity games?
  - Our recursive algorithm has complexity $\mathbf{O(mn^d)}$ for a game with $\mathbf{m}$ edges, $\mathbf{n}$ positions, $\mathbf{d}$ colours

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- Problem originally posed by Church/Büchi, solved by Büchi and Landweber in 1969
- Can be extended to certain kinds of infinite game graphs that are finitely generated
  - Pushdown graphs, corresponding to an automaton with a stack
- The model checking problem for modal μ-calculus directly reduces to solving parity games
- What is the complexity of constructing a memoryless winning strategy for parity games?
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- Can we do improve on LAR for winning conditions that require memory?

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