Lecture 13: From Streett automata to Rabin automata and Back

Rabin and Street conditions are "complements" of each other and so it is not obvious as to how to transform an automaton of one type into the other. Let A be a Streett automaton with accepting family $(E_1, F_1), (E_2, F_2) \ldots (E_k, F_k)$. A run ρ is accepting iff

$$\forall i.\ (\mathit{inf}(\rho) \cap F_i \neq \emptyset) \Rightarrow (\mathit{inf}(\rho) \cap E_i \neq \emptyset).$$

We keep track of the E_is and F_is hit along a run as part of the state. We then use this to translate the Street acceptance condition into an equivalent Rabin condition. The idea is similar to that used in translating Müller conditions to Rabin conditions via LARs. Instead of keeping a permutation of the states we keep a permutation of the indices $1, 2 \ldots k$ giving the order in which the sets $E_1, E_2, \ldots E_k$ were last seen. We shall use $\mathsf{Perm}(k)$ to refer to the set of permutations of $1, 2, \ldots k$. However, we need to do a little more work here because, in each move the run would visit a number E_i's (as opposed to single state) and all of them have to be moved to the right end of the sequence. As in the case of LARs we use a pointer e to keep track of the leftmost position from which a index was moved right in the last move.

With these ideas we set a state to be a triple (q, I, e) where q is a state of A and I is a permutation $(i_1, i_2, \ldots i_k)$ of $(1, 2, \ldots k)$ and $1 \leq e \leq k$. The transition relation is given by: $(q', I', e') \in \delta'((q, I, e), a)$ if I' is obtained from I by moving all the indices i with $q' \in E_i$ to the right end (in some fixed order, say the lexicographic order), and e' identifies the leftmost position p in I, such that such that $q' \in E_{i_p}$ (thus, e' identifies the leftmost position from which an index was moved right in the move leading to the current state). For example, consider a transtion $q \xrightarrow{a} q'$ where $q' \in E_1$ and $q' \in E_4$. From a IAR state $(q, (3, 2, 4, 5, 1), 2)$ we would get a transition on a to $(q', (3, 2, 5, 4, 1), 3)$. The last component is 3 since the left most position from which an index (4) was moved right is 3. To ensure that e is always defined we throw in the pair (Q, Q) to the set of Streett pairs (if it is not already there). Of course, this does not change the language accepted.

For any run ρ we write $\mathsf{Infl}(\rho)$ to denote the list of indices i such that the set E_i is visited infinitely often and $\mathsf{Finl}(\rho)$ to denote the indices i such that E_i is visited finitely often along ρ. In any infinite run

$$(q_0, I_0, e_0) \xrightarrow{a_1} (q_1, I_1, e_1) \xrightarrow{a_2} \ldots$$

of this automaton on a word $a_1 a_2 \ldots$, there is a point N beyond which all indices in $\mathsf{Infl}(\rho)$ appear to the right of all the indices in $\mathsf{Finl}(\rho)$. Let m be the leftmost position whose value is taken by e infinitely often along this run. Then, for all $j \geq N$, the first $m-1$ positions of I_j are identical and all of them belong to Finl. Moreover, if the index i appears among positions $m, m+1, \ldots k$ at I_j, $j \geq N$, then $i \in \mathsf{Infl}(\rho)$. So far everything has been pretty much as it was in the case of LARs, with states replaced by indices.

When is such a run accepting in A? The Streett acceptance requires that if F_i is hit infinitely often then the corresponding E_i must also be hit infinitely often. Equivalently, if F_i is hit infinitely often then i must appear among positions $m+1 \ldots k$ in $I_N, I_{N+1} \ldots$. To keep track of this we add another component, an index f, to the state. So, a state is a 4-tuple (q, I, e, f) where q,I and e are as before. f keeps track of the leftmost position p such that $q \in F_{i_p}$ where $I = (i_1, i_2, \ldots, i_k)$. (Notice the difference between e and f. e refers to the leftmost position in the *previous state* that was moved right in the last transition. f refers to a position in the current state.)

With this definition, a run $(q_0, I_0, e_0, f_0) \xrightarrow{a_1} (q_1, I_1, e_1, f_1) \ldots$ is accepting precisely when leftmost position that appears as e_j for infinitely many js is to the left of the leftmost position that appears as f_j for infinitely many js. Equivalently, the run is accepting precisely when there is a position p such that

1. For infinitely many j, $e_j = p$.
2. $f_j < p$ for only finitely many j.

This follows from the fact that if p is hit infinitely often then $p \geq m$ and item 2 guarantees that $f_j < m$ only finitely many times. Thus, beyond some point $f_j \geq m$ and thus if F_i is hit infinitely often then so is E_i. The two conditions above can be directly translated as a Rabin pair and thus we can translate any Streett automaton into an equivalent Rabin automaton with $O(n.r!)$ states and $O(r)$ accepting pairs.

Theorem 1 *Let $A = (Q, \Sigma, \delta, s, ((E_1, F_1), (E_2, F_2) \ldots (E_k, F_k))$ be a Streett automaton. Then the automaton*

$$\mathsf{IAR}(A) = (Q \times \mathsf{Perm}(k) \times \{1, \ldots k\} \times \{1, \ldots, k\}, \Sigma, \Delta, (s, (1, 2, 3, \ldots, k), k, k), ((E'_1, F'_1), \ldots (E'_k, F'_k))$$

where the transition relation Δ is as defined in the above discussion and

$$\begin{aligned} F'_i &= \{(q, I, e, f) \mid e = i\} \\ E'_i &= \{(q, I, e, f) \mid f < i\} \end{aligned}$$

accepts the same language as A. Further, A' is deterministic whenever A is deterministic. Thus any Streett automaton can be translated into and equivalent Rabin automaton, preserving determinism, whose size is bounded by $O(n.r!)$ and which has at the most r accepting pairs.

Can we extend this to a transformation from Streett automata to Parity/Rabin-Chain automata? Here is how: As discussed above a run ρ is accepting if the leftmost position that appears as e_j for infinitely many j is to the left of the leftmost position that appears as f_j for infinitely many j. This smacks of a parity condition! Note that e_j can take values among $1, 2, \ldots k$ and similarly for f_j. Let us assign the value $2i$ to e instead of i (so that e now takes values from the set $\{2, 4, \ldots 2k\}$) and the value $2i + 1$ to f instead of i (so that f takes values from the set $\{3, 5, \ldots, 2k + 1\}$).

Suppose that in the original run, the leftmost position taken infinitely often by e is j and that taken by f is l. With the new values the smallest value taken by e along this infinite run is $2j$, while the smallest value taken by f along the run is $2l + 1$. Notice that $j \leq l$ if and only if $2j < 2l + 1$. Equivalently, $j \leq l$ if and only if the smallest value taken infinitely often by either e or f is an even number. We use this to define the parity automaton. We do not bother to replace the values of e and f by $2e$ and $2f + 1$, but use the rank function to capture this.

Theorem 2 *Let* $A = (Q, \Sigma, \delta, s, ((E_1, F_1), (E_2, F_2) \dots (E_k, F_k))$ *be a Streett automaton. Consider the automaton*

$$\mathsf{IAR}(A) = (Q \times \mathsf{Perm}(k) \times \{1, 2, \dots k\} \times \{1, 2, \dots, k\}, \Sigma, \Delta, (s, (1, 2, 3, \dots, k), 2k, 2k+1), \sigma)$$

where the transition relation Δ *is as in theorem 1 and the rank function* σ *is given by*

$$\sigma((q, I, e, f)) \;=\; \mathsf{Min}(2e, 2f + 1)$$

This parity automaton accepts the same language as A, more over this automatonis deterministic whenver A is deterministic. The size blowup is bounded by $O(n.r!)$ *and the number of levels in the rank function is bounded by* $O(r)$.

Rabin to Streett

Recall that a Rabin automaton can be transformed into an equivalent Nondeterministic Büchi automaton of size $O(n.r)$. And since NBAs are also NSAs, this gives a efficient way to tranform NRAs into NSAs. But this does not preserve determinism. However, given a DRA, we can complement it to get a DSA (without any blowup), transform the DSA into an equivalent Deterministic Parity automaton (of size $O(n.r!)$ and with at the most $O(r)$ rank levels) and finally complement this parity automaton (without any blowup) to get a DPA automaton equivalent to the original Rabin automaton. Thus, we also have transformations from Rabin automata to Streett automata and Parity automata with a size blow up at at most $O(n.r!)$ and with at the most $O(r)$ size for the acceptance condition.

A Lowerbound on transformations from Streett Automata

In this section we shall show that the transformation from deterministic Streett automata to deterministic Rabin automata described in the previous section is optimal. This result (and the following proof) is due to Christof Löding ([**?**]).

The technique used to prove this result will be almost identical to that used to proving the optimality of Safra's construction. Recall, that the proof there proceeded using the following steps:(where L_n is the language used in that proof.)

1. Assume that there is a NSA of size $< n!$ that accepted the language L_n.
2. Pick words $\sigma_1, \sigma_2 \dots \sigma_k$ in L_n and corresponding accepting runs $\rho_1, \rho_2, \dots \rho_k$.
3. Construct a new word σ and a run ρ on σ by splicing together parts of $\rho_1, \rho_2, \dots$ such that $\mathit{inf}(\rho) = \bigcup_{1 \leq i \leq k} Inf(\rho_i)$.
4. Show that $\sigma \notin L_n$ and use the fact that if X and Y satisfy a Streett condition then so does $X \cup Y$ to conclude that we have an accepting run on a word not in L_n and arrive at a contradiction to 1.

A similar technique would not work for Nondeterministic Rabin automata. For Rabin conditions the analogue of the property used in item 3 above is: if X and Y do not satisfy a Rabin condition then $X \cup Y$ also does not satisfy the Rabin condition. To use this property we have dualize the entire argument and get:

1. Assume that there is a NRA of size $< n!$ that accepted the language L_n.
2. Pick words $\sigma_1, \sigma_2, \ldots \sigma_k$ outside L_n and corresponding non-accepting runs $\rho_1, \rho_2, \ldots \rho_k$.
3. Construct a new word σ and a run ρ on σ by splicing together parts of $\rho_1, \rho_2, \ldots$ such that $\mathit{inf}(\rho) = \bigcup_{1 \le i \le k} Inf(\rho_i)$.
4. Show that $\sigma \in L_n$ and use the fact that if X and Y do not satisfy a Rabin condition then $X \cup Y$ also does not satisfy it to conclude that ρ is not an accepting run.

But we have no contradiction since there might be other runs that accept σ. As a matter of fact, I do not know any technique to show a lowerbound on the translation from Streett automata to Nondeterministic Rabin automata.

However if we work with Deterministic Rabin automata the above technique works, because, ρ is the unique run for the automaton on σ and if it is rejecting then there is no accepting run and so it rejects a word in L_n and thereby contradicts 1. This is the scheme we shall follow here. Consider the family of automata $(A_n)_{n \ge 1}$ described below.

$$A_n = (\{i, -i \mid 1 \le i \le n\}, \{1, 2, \ldots n\}, \delta, s, ((\{1\}, \{-1\}), (\{2\}, \{-2\}), \ldots, (\{n\}, \{-n\})))$$

where $\delta(i, j) = -j$ and $\delta(-i, j) = j$ for all $1 \le i, j \le n$. The automaton is in negative states after even number of moves and in positive states after odd number of moves. Moreover, the state reached after reading a word w depends only on the last letter of w (and the parity of the length of w).

Let $\rho = a_0 a_1 \ldots$ be a word over Σ. The set of positive states entered during this run is completely determined by $a_0 a_2 a_4 \ldots$ and similarly the set of negative states entered is determined by $a_1 a_3 \ldots$. In particular if $\mathsf{even}(\rho)$ is the set of letters that appear infinitely often at positions $0, 2, \ldots$ and $\mathsf{odd}(\rho)$ is the set of letters that appear infinitely often at positions $1, 3, \ldots$ then, the word ρ is accepted if and only if $\mathsf{odd}(\rho) \subseteq \mathsf{even}(\rho)$. This leads us to the following observation:

Observation 1: Let u be a word of even length over Σ_n. For any ρ, ρ is in the language L_n if and only if $u\rho$ is in L_n.

The proof proceeds by induction on n. We shall actually prove a slightly stronger result. Namely:

The Hypothesis: Any deterministic Rabin automaton A accepting the language L_n must contain a strongly connected component with at least $n!$ states.

If n is 1, the result follows immediately as any automaton accepting L_1 must have at least a scc with one state.

For the induction step, pick an automaton A with minimum size that accepts L_n. If we omit the letter i from the alphabet of A, it accepts the language L_{n-1} (modulo some renaming of letters) and therefore, by the induction hypothesis, we have the following observation:

Observation 2: The automaton A has an scc with least $(n-1)!$ states. As a matter of fact, A restricted to the set of letters $\Sigma - \{i\}$ has a scc with at least $(n-1)!$ states.

We can say something interesting about the structure of A:

Observation 3: The automaton A considered as a graph is strongly connected.

Suppose A has multiple sccs. These sccs themselves form a directed acyclic graph. Pick any scc at the leaf of this DAG. Now, there is a word of even length u from s to some state q in this scc. But $\sigma \in L_n$ if and only if $u\sigma \in L_n$. Thus, the automaton A with q as the start state accepts exactly the same language as A. But the states reachable from q are only its scc (since this scc forms a leaf in the DAG on sccs). Thus, we might as well restrict ourselves to this scc and use q as the start state to obtain an automaton that accepts L_n. But we had assumed that A was the smallest automaton that accepts L_n. Therefore, A must consist of just a single strongly connected component.

Observation 4: Any automaton A accepting L_n has a sub scc that accepts L_n.

This just follows from the argument given above to prove Observation 3.

Now, for each i we construct a word σ_i that $\mathsf{even}(\sigma_i) = \Sigma_n - \{i\}$ and $\mathsf{odd}(\sigma_i) = \Sigma_n$. Thus none of these words are accepted by A. The word σ_i and the run of A on σ_i are constructed in unison as follows: Let the word $u_s^i = 112233\ldots(i-1)(i-1)(i+1)(i+1)\ldots nn$. This word has every letter in $\Sigma_n - \{i\}$ in both odd and even numbered positions. Suppose this word leads us from s to some state q. Now, A with q as starting state accepts L_n. Thus, by Observation 2, there is an scc of size at least $(n-1)!$ reachable from q even when the alphabet is restricted to $\Sigma_n - \{i\}$. Let w be word over $\Sigma_n - \{i\}$ that labels a path from q to this scc and visits at least $(n-1)!$ states in this scc. We set $v_s^i = u_s^i.w.ik$ where $k \in \Sigma_n - \{i\}$. We point out two important properties of this word:

1. The set of letters appearing at odd positions is Σ_n and the set of letters appearing at even numbered positions is $\Sigma_n - \{i\}$.

2. The run ρ_s^i of the automaton A on this word (starting at s) has atleast $(n-1)!$ distinct states.

We could do this for any state q. That is, we could construct a word v_q^i such that the two properties listed above are satisified (with q in place of s.) The word $\sigma_i = v_s^i v_{q_1}^i v_{q_2}^i \ldots$, where q_{i+1} is the state reached on reading v_q^i starting at state q_i. The run ρ_i on σ_i is $\rho_s^i \rho_{q_1}^i \ldots$.

We now show that $inf(\rho_i) \cap inf(\rho_j)$ must be empty whenever $i \neq j$. Suppose that a state q appears infinitely often in both these runs. Let $\rho_i = r_1.r_2.\rho'_i$ where

1. r_1 leads to the state q
2. r_2 (begins and) ends in the state q
3. r_2 includes at least one of the ρ^i_p entirely (and therefore the word read along r_2 contains some v^i_p entirely)
4. r_2 visits exactly the set of states in $inf(\rho_i)$.

This can always be ensured since q is visited infinitely often.

Similarly we can write $\rho_j = t_1.t_2.\rho'_j$ satisfying similar properties. Now consider the run $\rho = r_1(r_2t_2)^\omega$. This run reads a word σ that includes infinitely many copies of some ρ^i_p and infinitely many copies of some $\rho^j_{p'}$. Thus, $\mathsf{even}(\sigma) = \mathsf{odd}(\sigma) = \Sigma_n$. Thus, the word is in L_n. However, the set of states hit infinitely often along this run is $inf(\rho_i) \cup inf(\rho_j)$. Thus the run ρ cannot satisfy the Rabin condition. However, the automaton is deterministic and ρ is the only run of this automaton on σ and this contradicts the assumption that $L(A) = L_n$. Hence, we may conclude that $inf(\rho_i) \cap inf(\rho_j)$ is empty whenver $i \neq j$. Notice that there are at least $(n-1)!$ states in each $inf(\rho_i)$. Thus, there are at least $n!$ states in A.

Now, we can apply Observation 4 to conclude that any automaton accepting L_n has an scc with at least $n!$ states.

References

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