Moduli of π-Vector Bundles over an Algebraic Curve

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Introduction

Let X be a smooth algebraic curve, proper over the field $\mathbf{C}$ of complex numbers (or equivalently a compact Riemann surface) and of genus g. Let J be the Jacobian of X; it is a group variety of dimension g and its underlying set of points is the set of divisor classes (or equivalently isomorphic classes of line bundles) of degree zero.

It is a classical result that the underlying topological space of J can be identified with the set of *(unitary) characters* of the fundamental group $\pi_1(X)$ into $\mathbf{C}$ (i.e. homomorphisms of $\pi_1(X)$ into complex numbers of modulus one) and therefore $J = S^1 \times \cdots \times S^1$, g times, as a topological manifold, S^1 being the unit circle in the complex plane.

The purpose of these lectures is to show how this result can be extended to the case of unitary representations of arbitrary rank for Fuchsian groups with compact quotients.

Given a representation $\rho\colon \pi_1(X) \to GL(n,\ \mathbf{C})$, one can associate to ρ in a natural manner (as will be done formally later) a vector bundle (algebraic or holomorphic) on X; let us call such a vector bundle *unitary* if ρ is a unitary representation.

[1] This is essentially a reproduction of the paper with the same title that appeared as a part of the proceedings of a conference held in Italy in September 1969 (Questions on Algebraic Varieties C.I.M.E., III Ciclo, Varenna, 1969). Many typographical mistakes as well as a mathematical error (pointed out by V. Balaji, see Remark 6, Chapter II) in the earlier version have been corrected.

It is easy to show that two vector bundles V_1, V_2 associated to unitary representations ρ_1, ρ_2 are isomorphic (in the algebraic or holomorphic sense) if and only if ρ_1 and ρ_2 are equivalent as representations.

Suppose that the genus of X is ≥ 2. Recall the following results (cf. [12], [17]).

(1) The unitary bundles on X can be characterized *algebraically.*

To be more precise, let us call a vector bundle V on X of degree zero on X *stable* (respectively *semi-stable*) (this definition is due to Mumford) if for every proper sub-bundle W of V, one has $\deg W < 0$ (respectively $\deg W \leq 0$).

Then one has the following: a vector bundle V on X is unitary if and only if it is a direct sum of stable bundles (of degree 0) and V is stable if and only if the corresponding unitary representation is irreducible unitary.

(2) On the equivalence classes of unitary representations of a given rank of $\pi_1(X)$ (equivalently, on the isomorphic classes of unitary vector bundles on X of a given rank), there is a natural structure of a *normal projective variety.*

It can be shown that *there are stable bundles on X of arbitrary rank if* $g \geq 2$ and that in any case, they form a *Zariski open subset* of an algebraic family of vector bundles.

Let $\tilde{X}$ denote a simply connected covering of X. Then $\pi = \pi_1(X)$ can be identified with a proper discontinuous group of automorphisms acting *freely* on $\tilde{X}$. It is easily seen, since π *operates freely on* $\tilde{X}$, that the study of (holomorphic) vector bundles on X is equivalent to the study of π-*vector bundles on* $\tilde{X}$, a π-vector bundle on $\tilde{X}$ being a (holomorphic) vector bundle E on $\tilde{X}$ together with an action of π on E compatible with its action on $\tilde{X}$.

From this point of view the results (1) and (2) admit of generalizations and in fact in the following, we shall be concerned with the following more general situation:

Let $\tilde{X}$ be a simply connected Riemann surface and Γ a proper discontinuous group of automorphisms of $\tilde{X}$ such that, $Y = \tilde{X} \bmod \Gamma$ is *compact* (the action of Γ is *not* supposed to be free). It is well-known that Γ has a normal subgroup of finite index Γ_0 in Γ such that Γ_0 operates *freely* on $\tilde{X}$.

Let $X = \tilde{X} \bmod \Gamma_0$ and $\pi = \Gamma \bmod \Gamma_0$. Then there is a canonical action of π on X such that $Y = X \bmod \pi$.

It is easily seen that the study of Γ-vector bundles on $\hat{X}$ is equivalent to the study of π-vector bundles on X and thus the study of Γ-vector bundles on $\tilde{X}$ can be said to be an algebraic problem.

Given a representation ρ of Γ into $GL(r, C)$, there is a natural Γ-vector bundle on $\tilde{X}$ (of rank r) and consequently a π-vector bundle E on X associated to ρ.

Let us call a π-vector bundle E on X *π-unitary* (respectively *irreducible π-unitary*) if it is π-isomorphic (i.e. isomorphic in the category of π-vector bundles) to a π-vector bundle associated to a unitary (respectively irreducible) representation of Γ.

As in the case of a free action, if E_1, E_2 are two π-unitary vector bundles on X associated to unitary representations ρ_1, ρ_2 of Γ, then E_1 is π-isomorphic to E_2 if and only if the representations ρ_1, ρ_2 of Γ are equivalent.

Let us call a π-vector bundle E on X of degree zero, *π-semi-stable* if the underlying vector bundle is semi-stable (as defined above) and *π-stable* if for every proper π-sub-bundle W of E, we have deg $W < 0$.

Let us call two π-vector bundles E_1 and E_2 on X locally isomorphic at $x \in X$ if there is a neighbourhood U of x invariant under π_x, the isotopy group of π at x, such that the restrictions of E_1 and E_2 to U are π_x-isomorphic; let us call E_1 and E_2 *locally isomorphic* if they are locally isomorphic at every point of X (unlike the usual case, any two π-vector bundles of the same rank need not be locally isomorphic).

We can thus speak of the *local type* of a π-vector bundle on X.

With these definitions, these lectures are devoted to the proof of the following results generalizing (1) and (2) above:

Suppose that genus of $Y = \tilde{X} \bmod \Gamma$ is ≥ 2. Then

Theorem I *A π-vector bundle on X of degree zero is π-stable if and only if it is irreducible π-unitary (cf. Theorem 4, Chapter II).*

Theorem II *On the space of isomorphic classes of π-unitary vector bundles on X of a fixed local type τ, there is a natural structure of a normal projective variety (cf. Theorem 5, Chapter II).*

We shall now give a brief outline of the proofs of the above theorems.

One proves directly that a π-unitary (respectively irreducible π-unitary) vector bundle E on X is π-semi-stable (respectively π-stable) of degree zero; further given an analytic family $\{E_t\}_{t\in T}$ of π-semi-stable vector bundles on X parametrized by an analytic space T, the subset T_0 of points $t \in T$ such that E_t is irreducible π-unitary is *closed* in T. (cf. Proposition 10, Chapter II).

Then one shows that given an analytic family of π-vector bundles $\{E_t\}_{t\in T}$ on X the subset T_0 of T such that E_t is irreducible π-unitary is *open* in T (cf. Proposition 7, Chapter II).

For proving this, one proceeds as follows.

Let ρ be an irreducible unitary representation of Γ of rank r and E the associated π-bundle on X.

Then it is proved that if U is the real analytic space of all unitary representations of Γ of rank r, then U is *smooth* (i.e. a manifold) in a neighbourhood of ρ and the dimension of the topological manifold constituted by this smooth neighbourhood is

$$2\dim_{\mathbf{C}} H^1(X,\ \pi,\ E^* \otimes\ E) + \dim\ K_r - 1$$

where K_r-denotes the group of unitary matrices of rank r, E^* the dual vector bundle of E and $H^1(X,\ \pi,\ E^* \otimes\ E)$ denotes a naturally defined first cohomology group of $E^* \otimes\ E$ in the category of abelian π-sheaves on X. (cf. Th.3 and Cor. 2, Th.3, Chap. 1).

Since K_r acts by inner conjugation on U and we have a natural induced *free* action of $PK_r = K_r$ modulo its centre (consisting of scalar matrices) on U, we have a manifold V such that $\dim V = 2\dim_{\mathbf{C}} H^1(X,\ \pi,\ E^* \otimes E)$ and which represents a *nice local moduli of unitary representations* around the point ρ (cf. Cor. 1, Th.3, Chap. I).

On the other hand it is known, after the work of Kodaira–Spencer, that there is a *nice local moduli* D of vector bundles around the point E and which is a complex analytic manifold of complex dimension $\dim_{\mathbf{C}} H^1(X,\ \pi,\ E^* \otimes E)$; we obtain this not by using the work of Kodaira-Spencer but as a consequence of studying a suitable Quot scheme in the sense of Grothendieck and which is anyway required for the proof of Theorem II (cf. Cor., Prop. 6, Chap. II).

Thus we have

$$\dim_{\mathbf{C}} D = 2\dim_{\mathbf{R}} V$$

and this implies easily that T_0 is open in T as required above.

Thus to prove Theorem I, it suffices to produce a *connected family* parametrizing all π-stable vector bundles of a fixed local type, in fact we construct such a family which is a smooth (irreducible) variety (cf. proof of Theorem 5, Chapter II).

To prove Theorem II, one constructs an algebraic family of π-vector bundles on X parametrized by a smooth, irreducible, quasi-projective variety R^τ containing all π-semi-stable vector bundles of a fixed local type τ and an action of a reductive algebraic group H on R^τ such that the orbits in R^τ under H correspond precisely to isomorphic classes of π-vector bundles (cf. Proposition 6, Chapter II).

This is done by using the Quot schemes of Grothendieck. The real difficulty now starts, because, as Mumford and Nagata have shown, the orbit spaces under the action of an algebraic group need not exist in the category of algebraic schemes even in good cases.

Now one follows the ideas of Mumford (cf. [8]) by reducing this question to a problem of constructing orbit spaces for a product of Grassmannians under the diagonal action of the projective group and connects the stable (respectively semi-stable) vector bundles with the *stable* (respectively *semi-stable*) points of this product of Grassmannians under the action of the projective group (cf. Proposition 9 and Corollary 2, Proposition 9 as well as §3, Chapter II).

Then Theorem II follows from these considerations for the case of usual vector bundles and a suitable modification gives it for the case of π-vector bundles. The Theorems I and II above, in the particular case of a *free* action of Γ, are the main results of [12] and [17] respectively.

The proofs outlined above for the general case are substantially the same as in [12] and [17].

Some technical improvements upon the proofs of [12] and [17] are included here.

The fact that the morphism χ of Corollary 2, Proposition 9, Chapter II is *proper* is taken from [18] and this makes the proof of Theorem II more direct than in [17] (for the case of the usual vector bundles).

Another fact worth mentioning is the proof (due to S. Ramanan) given here that an irreducible unitary bundle is stable.

This is more direct than that of [12].

The problem of constructing moduli of π-vector bundles over an algebraic curve, was raised for the first time by André Weil in [19].

In fact, most of the material in §2, Chap. I is to be found in [19]; in the presentation of this material i.e.§I and §2, Chap. I, we have followed the exposition of Grothendieck [4] of this paper of Weil.

The existence of a quasi-projective moduli space for *stable bundles* was first proved by Mumford [8].

Chapter I

Unitary π-bundles

1. *Generalities on π-bundles*

Let X be a complex analytic space and π a *discontinuous* group of automorphisms of X i.e. π acts as a group of analytic automorphisms of X and satisfies

(i) for all $x \in X$, the isotropy group π_x at x is finite and(ii) there exists an open neighbourhood U_x of x such that $\pi_x U_x = U_x$ and $U_x \cap gU_x = \emptyset$ for $g \notin \pi_x$.

Then it can be shown that $Y = X/\pi$ ($p: X \to Y$ canonical map) has a natural structure of a complex analytic space (cf. [2]).

(In fact, we require this only for the case when X is a manifold of dimension one and in this case it is easy to see that Y is a manifold of dimension one and the image of the canonical map $\pi_x \to \mathrm{Aut}\, U_x$ is cyclic).

A *fibre space* $p: E \to X$ (or an X-*analytic space*) is called a π-*fibre space* (or a π-*analytic space*) over X if π operates on E and $p: E \to X$ is a π-morphism i.e. commutes with the operations of π.

We say that a sheaf $\mathcal{G}$ (of sets, groups, rings etc.) on X is a π-*sheaf* if the corresponding étale space over X associated to $\mathcal{G}$ is a π-fibre-space over X (note that the étale space acquires canonically a structure of an X-analytic space).

The definition of a π-sheaf can also be formulated by a procedure resembling a presheaf datum; we leave the details.

Now π sheaves on X (respectively of sets, groups, rings etc.) form a *category* under morphisms which commute with the action of π.

Given a Y-analytic space, $E \to Y$, the *base change* of $E \to Y$ by $p: X \to Y$, namely $E \times_Y X$ is a π-analytic space over X, induced by the canonical operation of π on $E \times_Y X \subset E \times X$, taking the trivial action of π on E.

We denote $E \times_Y X$ by $p^*(E)$.

If $E \to Y$ is a local isomorphism (i.e. E is étale over Y), then $E \times_Y X$ is étale over X.

From this we conclude easily that if $\mathcal{G}$ is a sheaf (respectively of sets, groups, rings, etc), then $p^*(\mathcal{G})$ has a natural structure of a π-sheaf over X (respectively of sets, groups, rings etc.)

Then $\mathcal{G} \mapsto p^*(\mathcal{G})$ defines a functor from the category of sheaves on Y (respectively of sets, groups, rings, etc.) into the category of π-sheaves on X (respectively of sets, groups, rings etc.)

We note that we have a canonical functor map $H^0(Y, \mathcal{G}) \to H^0(X, p^*(\mathcal{G}))$ (i.e. π-invariant sections of $p^*(\mathcal{G})$ over X).

For every π-sheaf $\mathcal{G}$ on X, we denote by $p_*(\mathcal{G})$, the *direct image* of $\mathcal{G}$ by p (sections of $p_*(\mathcal{G})$ on an open $V \subset Y$ are sections of $\mathcal{G}$ over $p^*(V)$).

We note that $p_*(\mathcal{G})$ acquires a natural structure of a π-sheaf on Y (taking the trivial action of π on Y).

We denote by $p_*^\pi(\mathcal{G})$ (the *invariant direct image of* $\mathcal{G}$) the subsheaf consisting of π invariants of $p_*(\mathcal{G})(p_*^\pi(\mathcal{G})$ is defined by the following presheaf: to every open V in Y assign the π-invariant sections of $\mathcal{G}$ over $p^*(V))$.

We note that $p_*^\pi(\mathcal{G})$ is a sheaf of groups, rings, etc. according as $\mathcal{G}$ is so.

Now $\mathcal{G} \mapsto p_*^\pi(\mathcal{G})$ defines a functor from the category of π-sheaves on X (respectively of sets, groups, rings) to the category of sheaves on Y (respectively of sets, groups, rings).

If we denote by $H^0(X,\ \pi,\ \mathcal{G})$ the set of π-invariant sections of $\mathcal{G}$ over X, we have $H^0(X,\ \pi,\ \mathcal{G}) = H^0(Y,\ p_*^\pi(\mathcal{G}))$. If E is a sheaf on Y, then $p_*^\pi(p^*(E))$ identifies canonically with E.

On the other hand, if $\mathcal{G}$ is a π-sheaf on X, then $p^*(p_*^\pi(\mathcal{G}))$ identifies with the subsheaf $\mathcal{G}^\pi$ of $\mathcal{G}$, whose stalk at x is $\mathcal{G}_x^{\pi_x}$-the subset of π_x-invariants of $\mathcal{G}_x$ under π_x.

Since $\mathcal{G} \mapsto p_X^\pi(\mathcal{G})$ and $E \mapsto p^*(E)$ are functors, we conclude easily that when π *operates freely*(so that $\pi_x = \mathrm{Id}.$ for every $x \in X$), the functor p_*^π establishes an equivalence of the category of π-sheaves (respectively of sets, groups, rings etc.) on X with the category of sheaves on Y (respectively of sets, groups, rings, etc.) (similarly for p^*).

We note that $\mathcal{O}_Y = p_*^\pi(\mathcal{O}_X)$, where $\mathcal{O}_X,\ \mathcal{O}_Y$ denote the structure sheaves of rings of X and Y respectively.

Similarly if G is a complex Lie group, then the sheaf $\mathcal{O}_X(G)$ of germs of analytic morphisms of X into G is a π-sheaf (of groups) and we have $p_*^\pi(\mathcal{O}_X(G)) = \mathcal{O}_Y(G)$.

Let $P \to X$ be a principal fibre space (analytic) with structure group a complex Lie group G.

We say that P is a π-*principal fibre space with structure group* G (or briefly a π-G bundle) if we are given an operation of π on P which commutes with that of G and induces the given operation of π on X.

We define in the obvious manner an isomorphism of two (π-G) bundles and denote the isomorphism classes by $H^1(X,\ \pi,\ \mathcal{O}_X(G))$.

We define similarly the notion of an associated π-bundle to a principal (π-G) bundle on X. We can thus speak of a π-vector bundle, namely the π-vector bundle associated to a π-$GL(n)$ bundle.

We have a natural notion of a π-*homomorphism* between two π-vector bundles and when we speak of the category of π-vector bundles, we take for morphisms π-homomorphisms.

Given a homomorphism $\rho\colon\pi \to G$ of π into a complex Lie group G, we obtain in a natural manner a π-G bundle $P \to X$ as follows: we take $P = X \times G$ and define the operation of π on P as $\alpha \circ (x,g) = (\alpha \circ x,\ \rho(\alpha)g), \alpha \in \pi$.

In particular if $\mathcal{G} = GL(V)$ (group of linear automorphisms of a finite-dimensional vector space V over $\mathbf{C}$), we get a π-$GL(V)$ bundle and its associated vector bundle will be referred to as the π-*vector bundle associated to the representation* ρ.

The direct sum (as well as tensor product) of two π-vector bundle has a natural structure of a π-vector bundle. The dual E^* of a π-vector bundle has a natural structure of a π-vector bundle. If $E_1,\ E_2$ are two π-vector bundles associated to representations $\rho_1,\ \rho_2$ of π, then the π-vector bundle $E_1 \otimes E_2$ is associated to the representation $\rho_1 \otimes \rho_2$.

(Similarly we get a statement for the dual representation).

Let $V_1,\ V_2$ be the representation spaces of $\rho_1,\ \rho_2$ respectively. Then we have a canonical homomorphism

$$\mathrm{Hom}_\pi(V_1,\ V_2) \to \mathrm{Hom}_\pi(E_1,\ E_2)$$

where $\mathrm{Hom}_\pi(V_1,\ V_2)$ denotes the $\mathbf{C}$-linear space of π-homomorphisms of V_1, into V_2 and $\mathrm{Hom}_\pi(E_1,\ E_2)$ denotes the $\mathbf{C}$-linear space of π-homomorphisms (analytic) of E_1 into E_2.

Proposition 1 *Let us suppose that X is a connected complex manifold and that Y is compact. Let $\rho_1,\ \rho_2$ be two unitary representations of π on (finite dimensional) vector spaces $V_1,\ V_2$ (i.e. $\rho_1,\ \rho_2$ leave invariant positive definite Hermitian forms on V_1, V_2 respectively). Then the canonical homomorphism*

$$\mathrm{Hom}_{\pi}(V_1,\ V_2) \to \mathrm{Hom}_{\pi}(E_1,\ E_2)$$

is an isomorphism.

In particular, if ρ is a unitary representation on V and E the associated π-bundle, then the natural map

$$V^{\pi} \to H^0(X,\ \pi,\ E)$$

is an isomorphism.

Proof. It suffices to prove the second assertion because of the following $\mathrm{Hom}_{\pi}(V_1,\ V_2) = \mathrm{Hom}_{\pi}(\mathbf{C},\ V_1^* \otimes V_2)$ (π operating trivially on $\mathbf{C}$) $= (V_1^* \otimes V_2))^{\pi}$.

Similarly $\mathrm{Hom}_{\pi}(E_1,\ E_2)$ is $H^0(X,\ \pi,\ E_1^* \otimes E_2)$.

Thus it suffices to prove that the canonical map $(V_1^* \otimes V_2) \to H^0(X,\ \pi,\ E_1^* \otimes E_2)$ is an isomorphism.

Now $E_1^*,\ E_1^* \otimes E_2$ are unitary bundles and thus we are reduced to proving the last assertion.

Now $E = X \times V$ and thus a π-section s of E can be identified with a holomorphic map $F{:}X \to V$

If $\|\ \|$ denotes a Hermitian metric on V invariant under ρ, we see that the function $h{:}X \to \mathbf{R},\ h(x) = \|F(x)\|^2$ is π-invariant.

This implies that h goes down to a function $g{:}Y \to \mathbf{R}$ and g is obviously continuous. In particular g attains both its maximum and minimum at some points of Y. This shows that h also attains its maximum and minimum at some points of X.

If we introduce a basis $\{e_i\}$ in V such that if $v = \sum z_i v_i,\ z_i \in \mathbf{C},\ \|v\|^2 = \sum |v_i|^2$, we have $F(x) = \sum F_i(x) e_i$ and $h(x) = \|F(x)\|^2 = \sum |F_i(x)|^2$ i.e. $h(x)$ is a sum of squares of moduli of holomorphic functions on X.

It follows that $h(x)$ is plurisubharmonic and since it attains its maximum at an interior point of the connected manifold $X,\ h$ reduces to a constant map.

From this one concludes that the holomorphic map $F{:}X \to V$ is a constant map because of the following:

Lemma 1 *Let X be a connected complex manifold and $h(x) = \sum_{i=1}^{r} |F_i(x)|^2$, where F_i is a holomorphic function on X. Then if h is constant, the F_i also reduce to constant functions.*

Proof. It is easy and left as an exercise.

Since $F{:}X \to V$ is a constant map, we see immediately that $F(X) \subset V^{\pi}$. From this it is immediate that the map $V^{\pi} \to H^0(X,\ \pi,\ E)$ is an isomorphism and the proposition is proved.

Corollary *Let $E_1,\ E_2$ be two π-vector bundles on X associated to unitary representation $\rho_1,\ \rho_2$ of π (X as in the proposition). Then E_1 is isomorphic to E_2 as π-bundles if and only if the representations ρ_1 and ρ_2 are equivalent.*

Proof. This is an immediate consequence of the proposition.

Let P be a $(\pi\text{-}G)$ bundle (G a complex Lie group) on X such that the underlying principal G-bundle is trivial i.e. $P \cong X \times G$.

We can write the operation of π on $X \times G$ as follows (taking the operation of G on P to be on the right):

$$(*) \qquad \alpha \cdot (x,g) = (\alpha \cdot x,\ f_\alpha(x)g)$$

where $f_\alpha{:}X \to G$ is a holomorphic map.

Writing down the conditions that π operates on $X \times G$, we obtain

$$f_{\alpha\beta}(x) = f_\alpha(\beta x) f_\beta(x), \quad \alpha, \beta \in \pi$$

i.e. $\alpha \mapsto f_\alpha$ defines a 1-*cocycle* of π with values in $\Gamma(X,\ \mathcal{O}_X(G))$, for the canonical operation of π on $\Gamma(X,\ \mathcal{O}_X(G))$ (note that $f(x) \mapsto f(\beta x),\ \beta \in \pi$, defines an operation of π on $\Gamma(X,\ \mathcal{O}_X(G))$, on the right).

Conversely, given a 1-cocycle of π with values in $\Gamma(X,\ \mathcal{O}_X(G))$ we get an operation of π on $P = X \times G$ by $(*)$, commuting with the action of G and thus P acquires a π-bundle structure.

Two such $(\pi\text{-}G)$ bundles $P_1$and P_2 given by 1-cocycles $\{f_\alpha(x)\}$ and $\{g_\alpha(x)\}$, are π-isomorphic if and only if the cocycles are cohomologous. i.e. there exists a holomorphic map $F{:}X \to G$ such that $F(\alpha x) = g_\alpha(x)F(x)f_\alpha(x)^{-1}$; also the set of isomorphic classes of such bundles can be identified with the set $H^1(\pi,\ \Gamma(X,\ \mathcal{O}_X(G))$ (this has no structure of a group if G is non-abelian).

We see that π-bundles associated to representations of π are particular cases of these π-bundles.

Given any $(\pi\text{-}G)$bundle, we can choose a neighbourhood U_x for every $x \in X$ invariant under π_x such that the restriction of the underlying G-bundle to U_x is trivial and thus *locally every $(\pi\text{-}G)$ bundle is of this type*.

Suppose now F is a *coherent* π-sheaf on X, then $p_*^\pi(F)$ is a coherent sheaf on Y. (Choosing neighbourhood U_x of x invariant under π_x, it suffices to show that $p_*^\pi x\ (F/U_x)$ is coherent on U_x/π_x and thus to show coherence of $p_*^\pi(F)$, we are reduced to the case of a finite group. Further this fact is immediate when X is a manifold of dimension one.)

Suppose further that Y is *compact*, then since $H^0(X,\ \pi,\ F) = H^0(Y,\ p_*^\pi(F))$, it follows that $H^0(X,\ \pi,\ F)$ is finite-dimensional.

We say that a π-vector bundle E on X is π-*indecomposable* if whenever $E = E_1 \oplus E_2$ as π-bundles, it follows that $E \cong E_1$ or E_2 (as π-bundles). Every π-vector bundle E on X can be written as a direct sum of indecomposable π-vector bundles E_i.

If moreover $Y = X/\pi$ is *compact then the E_i as well the "multiplicity" with which E_i occurs in E are determined uniquely* The proof is exactly the same as in the case of vector bundles on a compact complex manifold.

If A is the ring of π-endomorphisms of E, then $A = H^0(X,\ \pi,\ E^* \otimes E),\ (E^* =$ dual of $E)$ is finite-dimensional. In particular, A is artinian. A decomposition of E into π-indecomposable components is equivalent to a decomposition of the identity element of A into mutually orthogonal "indecomposable" idempotents. We get the proof by applying the usual Krull-Remak-Schmidt theorem to A considered a module over itself.

Let $\mathfrak{m}_X(G)$ be the sheaf of germs of *meromorphic* maps of X into a complex linear group G (for example if $G = GL(n)$, a meromorphic map is a matrix whose entries are meromorphic functions on X such that, on a dense open subset of X it defines a holomorphic map into G). Now $\mathcal{O}_X(G)$ is a subsheaf of groups of $m_X(G)$. The quotient sheaf of sets $\mathcal{O}_X(G)\backslash m_x(G)$ (operation on the left by $\mathcal{O}_X(G)$) is called the *sheaf of germs of divisors with values in G (or G-divisors)* and denoted by $D_X(G)$.

Now π operates on $D_X(G)$ and so $D_X(G)$ becomes a π-sheaf. A π-invariant section of $D_X(G)$ is called a $(\pi$-$G)$*divisor*. Given a $(\pi$-$G)$ divisor Θ and a point $x \in X$, there exists a π-invariant open subset U containing x such that $\Theta|U$, which is a π-invariant section of $\mathcal{O}_X(G)\backslash m_X(G)$ restricted to U, comes from a section of $m_X(G)|U$ i.e., a meromorphic map of U into G (this need *not* be π-invariant).

From this it follows easily that a $(\pi$-$G)$ divisor can be defined by a datum: an open covering $\{U_i\}$ of X by π-invariant open subsets and

$\{f_i\}$, where f_i is a meromorphic map of U_i into G such that $f_i(s^{-1}x) = \lambda_i^s(x) f_i(x)$, $\forall s \in \pi$, $\lambda_i^s(x)$ being a holomorphic map of U_i into G.

Now the transition functions $f_{ij}(x) = f_i(x) f_j^{-1}(x)$ which are holomorphic maps of $U_i \cap U_j$ into G define a G-principal bundle on X and through $\lambda_i^s(x)$ we can define an operation of π on this bundle.

Thus to a $(\pi\text{-}G)$ divisor we can associate a $\pi - G$-bundle (which is determined only up to a π-isomorphism).

The group of π-invariant meromorphic maps of X into G operates on the set of $(\pi\text{-}G)$ divisors on the *right* and we say that two $(\pi\text{-}G)$divisor Θ_1, Θ_2 are *equivalent* $(\Theta_1 \sim \Theta_2)$ if there exists a π-invariant meromorphic map F of X into G such that $\Theta_1 F = \Theta_2$ and it is easily seen that the $(\pi\text{-}G)$bundles defined by Θ_1 and Θ_2 are π-isomorphic if and only if $\Theta_1 \sim \Theta_2$.

If P is a $(\pi\text{-}G)$ bundle associated to a $(\pi\text{-}G)$divisor Θ, then Θ can be identified with a π-invariant meromorphic section of P. Suppose now that $G = GL(n)$ and that E is the associated vector bundle to P. Then Θ is defined by n π-invariant meromorphic sections of E, which are holomorphic and linearly independent in a dense open subset of X.

In general such a section need not exist and a $(\pi\text{-}G)$ bundle need not be defined by a divisor.

Let A be the abelian category of $\mathcal{O}_X$-modules. It has sufficiently many injectives. Let $\Gamma_X^G : A \to$ (Category of Abelian Groups), be the functor $F \mapsto H^0(X,\ \pi, F)$, $F \in A$. Then Grothendieck's theory applies and we define:

$$H^n(X,\ \pi, F) = R^n\, \Gamma_X^G \text{ (n-th right derived functor of Γ_X^G).}$$

Then given a short exact sequence in A, we get the familiar long exact sequence involving $H^n(X,\ \pi, F)$. We have indeed that $H^n(X,\ \pi, F)$ *is isomorphic to* $H^n(Y,\ p_*^\pi(F))$. For all these questions see Chap. V of [5].

2. π-vector bundles in the case of manifolds of dimension one

We shall suppose hereafter that X is a *connected* complex *manifold* of *dimension one* where π operates *faithfully* and that X as well as $X \bmod \pi$ are Hausdorff.

Then π_x is a cyclic group, say of order n_x, and we can take U_x to be isomorphic to a disc $D = \{z : |z| < r\}$ such that the operation of π_x is defined as: $\alpha \circ z = \zeta z$ where ζ is a primitive n_xth root of unity. Then $X \bmod \pi = Y$ is a manifold of dimension one.

If $x_1,\ x_2 \in X$ such that $p(x_1) = p(x_2)$, then π_{x_1} and π_{x_2} are conjugate subgroups of π and therefore the function $x \mapsto n_x$ is π-invariant and therefore gives rise to a positive integral valued function $y \mapsto n_y$ on Y. Now $n_y = 1$ for all but a *discrete* subset of Y. Such a function is called a *signature* on Y and the above function is called the *signature of* $p : X \to Y$.

The points $x \in X$ (respectively $y \in Y$) such that $n_x > 1$ (respectively $n_y > 1$) are called the *ramification points in* X (respectively Y) of $p{:}X \to Y$.

Given a signature $\{n_y\}$ on Y, it can be shown that there exists always a $p{:}X \to Y$ as above with the signature $\{n_y\}$ and in fact that there is a *unique* (upto Y-isomorphism) *"maximal" simply-connected* one with signature $\{n_y\}$, except for the case when Y is the Riemann sphere and $Y_0 = \{y \mid n_y > 1\}$ reduces to one point or two points $y_1,\ y_2$ with $n_{y_1} \neq n_{y_2}$.

If $\mathcal{F}$ is a coherent (π-$\mathcal{O}_X$)-module on X, *locally free* of rank n, then $p_*^{\pi}(\mathcal{F})$ is a coherent $\mathcal{O}_Y$-module *locally free* of n (that $p_*^{\pi}(\mathcal{F})$ is locally free is immediate since it is without torsion and coherent. Further it is seen immediately that $p_*^{\pi}(\mathcal{F})$ is of rank n in $Y - Y_0$, where $Y_0 = \{y \mid n_y > 1\}$. This implies that $p_*^{\pi}(\mathcal{F})$ is of rank n everywhere). It is well-known that we can find n meromorphic sections of $p_*^{\pi}(\mathcal{F})$ which are holomorphic and linearly independent at least at one point of Y.

Thus, if F denotes the π-vector bundle on X defined by $\mathcal{F}$, we conclude that F can be defined by a (π-$GL(n)$) divisor (*a* π-$GL(n)$ *divisor will be called simply a* π*-divisor for the sake of brevity)*. Thus every π-vector bundle on X can be defined by a π-divisor.

Let $\mathcal{F}$ be a coherent (π-$\mathcal{O}_X$)-module. Then we have $H^i(X,\ \pi, \mathcal{F}) = 0$ for $i \geq 2$, since we have $H^i(X,\ \pi, \mathcal{F}) = H^i(Y,\ p_*^{\pi}(\mathcal{F}))$ and $p_*^{\pi}(\mathcal{F})$ is a coherent $\mathcal{O}_X$-module.

Proposition 2 *Given a* π*-vector bundle* E *on* X *(of rank* n*) and* $x \in X$*, a* π_x*-invariant open neighbourhood* U_x *of* X *such that* E/U_x *can be defined by a representation* $\rho{:}\pi_x \to GL(n)$ *and* ρ *is determined uniquely in its equivalence class of representations (i.e. locally at* x, π*-vector bundles are classified by representations of* π_x *into* $GL(n)$*).*

Proof. Let $A = (\mathcal{O}_X(G))_x,\ G = GL(n)$. Now E is determined locally at x by an element of $H^1(\pi_x, A)$. We have a canonical map

$$\chi : H^1(\pi_x,\, GL(n)) \to H^1(\pi_x,\, A).$$

We have to show that χ is bijective.

Given a cocycle $\alpha \mapsto f_\alpha,\ f_\alpha \in (\mathcal{O}_X(G))_x$ evaluation at x i.e. taking the value of f_α at x defines a map

$$j : H^1(\pi_x,\, A) \to H^1(\pi_x,\, GL(n)).$$

We see that $j \circ \chi =$ identity. Thus is suffices to show that χ is *surjective*.

Now E is defined locally by a π-divisor. This means that if $\alpha \mapsto f_\alpha(x)$ is a 1-cocycle representing an element of $H^1(\pi,\, A)$, there exists $\Theta \in GL(n,\, K_x),\ K_x =$ quotient field of $\mathcal{O}_{X,x}$ such that

$$\Theta(\alpha \circ z) = f_\alpha(z)\Theta(z), \quad \alpha \in \pi_x,\ z \text{ in a neighbourhood of } x.$$

Let $\varphi_\alpha = f_\alpha(x)$; then φ_α defines a representation of π_x into $GL(n)$. Set

$$\psi(z) = \sum_{\alpha \in \pi_x} \varphi_\alpha \Theta(\alpha^{-1} z).$$

We have

$$\psi(\beta z) = \sum_{\alpha \in \pi_X} \varphi_\alpha \Theta(\alpha^{-1}\beta z) = \sum_{\alpha \in \pi_x} \varphi_\beta \varphi_{\gamma^{-i}} \Theta(\gamma z)$$

(setting $\alpha^{-1}\beta = \gamma$)

$$= \varphi_\beta \sum \varphi_{\gamma^{-1}} \Theta(\gamma z) = \varphi_\beta \cdot \psi(z).$$

i.e.

$$\psi(\beta z) = \varphi_\beta \psi(z).$$

Further, we have

$$\psi(z)\Theta^{-1}(z) = \sum_{\alpha \in \pi_x} \varphi_\alpha f_{\alpha^{-1}}(z)$$

since

$$\Theta(\alpha^{-1} z) = f_{\alpha^{-1}}(z)\Theta(z).$$

Since $f_{\alpha^{-1}}(x) = (\varphi_\alpha)^{-1}$, we get $\psi(x)\Theta^{-1}(x) = n_x \cdot$ Id.

This shows that $\psi(z)\Theta^{-1}(z)$ defines an element of $(\mathcal{O}_X(G))_x$, which in turn implies that $\psi(z)\Theta(z)$ defines the same divisor locally at x.

But the π_x-$GL(n)$ bundle defined by ψ is defined by the representation φ_α and this proves that χ is surjective. This proves the proposition.

Remark 1 Let E be a π-vector bundle on X of rank r and $x_1,\ x_2 \in X$ such that $p(x_1) = p(x_2)$. Now π_{x_1} and π_{x_2} are conjugate subgroups of π; choose an isomorphism of π_{x_2} onto π_{x_1} by one such conjugation.

Then by the above proposition E is defined locally at x_1 and x_2 by conjugate (or equivalent) representations of π_{x_1}.

We say that E is *locally of type* τ, where τ represents representations $\rho_i : \pi_{x_i} \to GL(r),\ x_i$ being a point of X chosen over every ramification point $y_i \in Y$ of $p : X \to Y$ if at $x_i,\ E$ is locally π_{x_i} isomorphic to the π_{x_i}-vector bundle defined by ρ_i.

All π-vector bundles of the same local type τ are mutually locally isomorphic (at every point of X).

Remark 2 The above proposition remains valid even in the case when X is a higher dimensional variety.

For the above proof to go through, we have only to show that every π-vector bundle E on X (of rank r) can be defined *locally* by a π-divisor. For this we note that for a suitably chosen π_x invariant neighbourhood U_x of $x,\ V = U_x/\pi_x$ is a Stein space.

Let E be the coherent sheaf associated to E; then one sees that the coherent sheaf $p_*^\pi(E) = F$ is locally free of rank r outside an analytic subset of V. Then from the fact that V is a Stein space, one concludes that F has n-sections which are linearly independent in a non-empty open subset of V.

This implies that E can be defined locally by a π-divisor.

Another way of proving this is to show that $H^1(\pi_x,\ K_x)$ is trivial (K_x quotient field of $\mathcal{O}_{X,x}$), which is again well-known.

Remark 3 Let $x \in X$, then $H^1(\pi_x,\ GL(r))$ which is the equivalence class of representations of π_x into $GL(r)$ can be identified with the set of all diagonal representations (π_x being cyclic of finite order) ρ of the form

$$\rho(\alpha) = \begin{pmatrix} \zeta^{d_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{d_r} \end{pmatrix}$$

where α is a generator of π_x, ζ is the primitive n_xth root of unity defined by $\alpha \cdot z = \zeta \cdot z$ (z a local coordinate at x) and $0 \le d_1 \le \cdots \le d_r < n_x - 1$.

Given a π-bundle E of rank r on X if $\rho : \pi_x \to GL(r)$ is the representation defining E locally at x, we see that if we normalize d_i as above, the d_i's are well-determined (i.e. independent of the local coordinates as well as the choice of α).

We see that if Δ is the matrix

$$\Delta = \begin{pmatrix} z^{d_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & z^{d_r} \end{pmatrix}, \quad z \text{ a local coordinate at } x$$

then every π-divisor defining Θ is of the form

$$\Delta \cdot \Theta_0$$

where Θ_0 is invariant under π_x i.e. entries in Θ_0 are meromorphic functions in $w = z^{n_x}$, which can be identified with a local coordinate at $y = p(x)$.

Remark 4 A π-divisor associated to a π-vector bundle E on X can be described purely in terms of Y. For every y, let A_y denote $\mathcal{O}_{Y,y}$ if $n_y = 0$ and if $n_y > 1$ denote the power series ring in w^{1/n_y}, w a local coordinate at y. Let L_y denote the quotient field of A_y.

Then a π-divisor associated to E is a map $y \mapsto [\Theta_y]$, where $\Theta \in GL(r, L_y)$ and $[\Theta_y]$ denotes the coset in $GL(r, A_y) \backslash GL(r, L_y)$ determined by Θ_y such that (i) for all but a discrete subset of Y, $\Theta_y \in GL(r, A_y)$ (ii) if $n_y > 1$,

$$\Theta_y = \begin{pmatrix} w^{d_1/n_y} & & 0 \\ & \ddots & \\ 0 & & w^{d_r/n_y} \end{pmatrix} (\Theta_0)_y, \quad 0 \le d_1 \le \cdots \le d_n < n_y.$$

where $\Theta_0 \in GL(r,\ K_y)$, K_y the quotient field of $\mathcal{O}_{Y,y}$.

The π-divisor is denoted often by Θ for shortness.

We recall that if Θ_1, Θ_2 represent π-divisors associated to π-bundles E_1 and E_2 of rank r then E_1 and E_2 are isomorphic if and only if there exists an $F \in GL(r,\ M)$, M being the field of meromorphic functions on Y, such that $\Theta_1 \cdot F = \Theta_2$.

Remark 5 Let Θ be a π-divisor on Y representing a π-vector bundle E of rank r on X as above. Then a π-invariant section of E on $p^{-1}(U)$, U open in Y, can be identified with a column matrix

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \cdot \\ f_1 \end{pmatrix}$$

such that the entries in $\Theta_y \mathbf{f}$ are in A_y for every $y \in U$. If y is such that $n_y > 1$, we see that $\Theta_y \mathbf{f}$ has elements in $A_y \Leftrightarrow (\Theta_0)_y \mathbf{f}$ has elements in $\mathcal{O}_{Y,y}$ (since $0 \leq d_i/n_y < 1$).

From this we conclude immediately that the vector bundle $p_*^\pi(E)$ on Y can be defined by the $GL(r)$-divisor Φ such that $\Phi_y = \Theta_y$ if $n_y = 1$ and $\Phi_y = (\Theta_0)_y$ if $n_y > 1$.

We now define the π-degree of a π-divisor Θ in the case Y is *compact*, as follows:

$$\pi\text{-}\deg \Theta = \sum_{y \in Y} \text{order } (\det \Theta_y)$$

($\det \Theta_y \in L_y$ and for an element $f \in L_y$, $f \neq 0$, order f denotes the rational number $\frac{p}{n_y}$, where p is the multiplicity of zero or pole of f at 0 considered as a function of $z = w^{1/n_y}$).

We define the *degree of a π-vector bundle E* as the degree of a π-divisor Θ on Y representing E. We see that this is well defined.

Taking Θ as above, we find

$$\pi\text{-}\deg E = \deg p_*^\pi(E) + \sum_{y, n_y > 1} \frac{d_1 + d_2 + \cdots + d_r}{n_y}, \ (d_i \text{ depend on } y).$$

$$= \pi\text{-}\deg \wedge^r E.$$

We note that if π is a finite group i.e. when X is *compact*,

$$\pi\text{-}\deg E = \frac{1}{\operatorname{ord} \pi} \cdot \deg E$$

(where $\deg E =$ degree of the line bundle $\wedge^r E$).

Proposition 3 *Let E be a $(\pi\text{-}G)$ bundle defined by a representation $\rho : \pi \to GL(r) = G$, and Y be compact. Then*

$$\pi\text{-}\deg E = 0.$$

Proof. There exists a π-invariant meromorphic section of E i.e. a π-meromorphic map $F\colon \to GL(r)$ such that

$$F(\alpha \cdot z) = \rho(\alpha) F(z),$$

and F gives rise to a divisor defining E. Let $f = \det F$. Then $f(\alpha \cdot z) = (\det \rho(\alpha)) \cdot f(z)$. Now

$$g = df/f$$

defines a π-invariant meromorphic differential on X and therefore defines canonically a meromorphic differential g' and Y. Now the sum of the residues at the poles of g' is zero. Now one checks that the residue of g' at y is precisely the order of f i.e. the order of $\det F$ at y in the sense defined above or more precisely the order of the divisor defined by Θ at y.

We conclude then, that

$$\sum_{y \in Y} \text{order of } F \text{ at } y = 0.$$

This proves the proposition.

We have now the basic

Theorem 1 (Weil) *Let X, Y be as above and suppose further that Y is compact and X simply connected. Let E be a π-vector bundle on X and E_i its π-indecomposable components. Then E is defined by a linear representation of π if and only if $\pi\text{-}\deg E_i = 0$ for all i.*

We will not prove this theorem here. For a proof see [19] or [4].

Let K_X denote the line bundle associated to the sheaf of germs of holomorphic 1-forms on X. Then K_X is canonically a π-line bundle. Then we have

Theorem 2 (Duality theorem) *Let E be a (holomorphic) π-vector bundle on X and E^* denote its dual π-vector bundle. Suppose that Y is compact. Then $H^1(X, \pi, E)$ can be identified canonically with the dual of the finite-dimensional vector space $H^0(X, \pi, E^* \otimes K_X)$.*

Proof. This is an immediate consequence of the usual duality theorem (on Y) because of

(i) $p_*^\pi(K_X) = K_Y$, $\quad p_*^\pi(E^* \otimes K_X) = (p_*^\pi(E))^* \otimes K_Y$ where K_Y denotes the line bundle on Y associated to the sheaf of germs holomorphic 1-forms on Y and

(ii) $H^i(X,\pi,F) = H^i(Y, p_*^\pi(F))$ where F is a coherent π-sheaf on X (we have mentioned (ii) towards the end of §1).

Let for every $y \in Y$, τ_y represent a character of π_x for some fixed choice of $x \in X$ such that $p(x) = y$ (the only non-trivial case is when $n_y > 1$) and $\tau = \{\tau_y\}_{y \in Y}$.

We say that a π-vector bundle E of rank r on X is τ-*special*, if for every $y \in Y, E$ is defined in a neighborhood of x, where x is the point chosen over y, by the representation $\rho : \pi_x \to GL(r)$, $\rho = \tau_y$. Id (Id = identity element of $GL(r)$).

Let $\mathcal{B}(\tau)$ represent the category of τ-special vector bundles, τ being fixed.

Then we have

Proposition 4 *Let $E_1, E_2 \in \mathcal{B}(\tau)$. Then the canonical homomorphism*

$$\chi\colon \mathrm{Hom}\,(E_1,E_2) \to \mathrm{Hom}\,(p_*^\pi(E_1),\ p_*^\pi(E_2))$$

is an isomorphism (i.e. the functor $E \to p_*^\pi(E)$ from the category $\mathcal{B}(\tau)$ to the category of vector bundles on Y is fully faithful).

Proof. That χ is injective is immediate. We have only to show that χ is surjective.

Given $y \in Y$, choice $x \in X$ over y given by the definition of τ. Choose a π_x-invariant neighbourhood of x such that the operation of π_x on U_x is given by $\alpha \cdot z = \zeta z$, where ζ is an n_xth root of 1 (n_x-the integral valued function on X defined before) and α is a generator of π_x.

Then for $\tau_y\colon \pi_x \to \mathbf{C}$, we have $\tau_y(\alpha) = \zeta^d$ for some $d \geq 0$.

If $E_1 \in \mathcal{B}(\tau)$, the associated coherent sheaf to E_1 can be represented as a free module M over $\mathcal{O}_{X,x}$ with basis say $e_1,\ldots,e_r$ such that if $m \in M, m = \sum_{i=1}^r f_i(z)e_i$, $f_i \in \mathcal{O}_{X,x}$

$$\alpha \cdot m = \sum_{i=1}^r \zeta^{-d} f_i(\zeta z).$$

One finds that $m \in M^\pi$, $m = \sum_i z^d f_i(z) e_i$, $f_i \in \mathcal{O}_{Y,y}$. This implies that the $\mathcal{O}_{X,x}$-submodule of M generated by M is $z^d \cdot M$.

Let N be a free $\mathcal{O}_{X,x}$-module representing E_2 in the same manner as M_1 represents E_1. Then we find that an element of $\mathrm{Hom}_{\mathcal{O}_{Y,y}}(M,N)$ extends to an $\mathcal{O}_{X,x}$-homomorphism of the submodule $z^d M$ into the submodule $z^d N$, i.e. in fact it extends to an $\mathcal{O}_{X,x}$-homomorphism of M into N.

This homomorphism is obviously π-invariant. This shows that an element of $\mathrm{Hom}\,(p_*^\pi(E_1)\ p_*^\pi(E_2))$ extends to a π-invariant homomorphism of E_1 into E_2 and the surjectivity of χ follows. This proves the proposition.

Proposition 5 *Let Y be a compact Riemann surface with genus $g \geq 1$ and y_0 a fixed point of Y. Let $\{n_y\}$ be the signature on Y such that $n_{y_0} = n$ and $n_y = 1$ for $y \neq y_0$. Let $p : X \to Y$ be the simply connected Riemann surface with signature $\{n_y\}$. Let x_0 be a point ever y_0 and τ the character of π_{x_0} such that $\tau(\alpha) = \zeta^q,\ n > q \geq 0,\ \alpha$ a generator of π_{x_0} such that $\alpha \cdot z = \zeta \cdot z,\ z$ local coordinate at x_0. Let F be a vector bundle on Y of rank n and degree $-q$.*

Then there exists a τ-special bundle E on X such that (i) $p_*^\pi(E) = F$ and (ii) E is associated to a representation of π.

Proof. Let Φ be a divisor $y \mapsto \Phi_y,\ y \in Y$ defining F. We now define a π-divisor Θ by defining $\Theta_y = \Phi_y,\ y \neq y_0,\ y \in Y$ and $\Theta_y = \Delta \cdot \Phi_y$ where $\Delta = z^{q/n} \cdot$ Id. (Id denoting identity matrix of order n).

Let E be a $(\pi\text{-}G)$ bundle defined by Θ Then as we have seen before $p_*^\pi(E)$ is isomorphic to F

Further E is indecomposable since E is so. We see that

$$\pi\text{-}\deg E = \deg F + \frac{1}{n} \cdot nq = 0$$

Therefore by Theorem 1, Chap. I, we conclude that E is associated to a representation of π, q.e.d.

Remark 6 Given a vector bundle F on a compact Riemann surface Y of rank n, we can find a line bundle L such that

$$-n < \deg(F \otimes L) \leq 0.$$

Thus in view of Propositions 4 and 5, the *study of vector bundles on Y of arbitrary degree can be reduced to the study of π-bundles defined by representations*.

If the degree is divisible by the rank, the theory of π-bundles when π-operates freely suffices but otherwise we need the case when π does not operate freely.

Proposition 6 *Let X be a simply-connected Riemann surface and π a (faithful) discontinuous group of automorphisms of X. Then there exists a normal subgroup π_0 of π and of finite index in π, such that π_0 operates freely on X.*

Proof. It is classical that X is either the Riemann sphere, the plane or the upper half plane. In all these cases the group of automorphisms of X is a group of matrices. The proposition is now an immediate consequence of the following.

Lemma 2 (Selberg) *Let M be a finitely generated group of matrices. Then there exists a normal subgroup M_0 of M of finite index such that M_0 does not contain any element of finite order.*

For a proof see [13].

Remark 7 Let X be as in Proposition 6 and π a discontinuous group of automorphisms such that $Y = X \bmod \pi$ is compact (Hausdorff).

Choose π_0 as in the proposition and let $X_1 = X \bmod \pi_0$. Let $p{:}X \to Y$, $q{:}X \to X_1$ and $p_1 : X_1 \to Y$ be the canonical maps. Let E be a π-vector bundle on X. Then $q_*^{\pi_0}(E)$ has a natural structure of Γ-bundle where $\Gamma = \pi/\pi_0$.

Now $q_*^{\pi_0}$ is an equivalence of categories since π_0 operates freely. This implies that the study of π-bundles on X is reduced to the study of Γ-bundles on X_1. But now X_1 is a compact Riemann surface and Γ is a finite group.

Now X_1 and Y have natural structures of algebraic schemes (smooth and projective over $\mathbf{C}$) and Γ is a group of automorphisms of this algebraic structure.

Besides, a holomorphic Γ-vector bundle on X_1 becomes an algebraic Γ-vector bundle for this structure and the algebraic classification and the holomorphic classification of these bundles coincide (cf. [15]). This reduces the study of π-bundles on X to an algebraic problem.

3. Manifold of irreducible unitary representations of π

Let X be the simply-connected Riemann surface over Y with the signature n_y (we suppose that we have chosen the signature such that X exists) and π the discontinuous group operating on X such that $Y = X/\pi$.

Let g be the genus of Y. Then it is a classical result that π can be identified with the group on the letters $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$ subjected to the relations

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_m = \mathrm{Id}.$$
$$C_1^{n_1} = C_2^{n_2} = \cdots = C_m^{n_m} = \mathrm{Id}.$$

Proposition 7 *Let ρ be a representation on a vector space V (over $\mathbf{R}$) such that $d = \dim V$ such that ρ is unitary (or more generally leaving invariant a non-degenerate bilinear form on V). Then we have*

$$\dim_{\mathbf{R}} H^1(\pi,\rho) = 2d(g-1) + 2\dim_{\mathbf{R}} H^0(\pi,\rho) + \sum_{\nu=1}^{m} e_\nu$$

where e_ν is the rank of the endomorphism $(\mathrm{Id} - \rho(C_\nu))$ of V. [$H^i(\pi,\ \rho)$ denotes the i^{th} cohomology group of π in V for the action through ρ].

Proof. Let us first indicate a proof for the case when the signature is trivial i.e. π operates freely on X (see [11]). Then the representation $\rho\colon \pi \to \mathrm{Aut}\, V$ defines a *local system* L on Y (i.e. a locally constant sheaf) of d-dimensional vector spaces. Then we have

$$\dim_{\mathbf{R}} H^0(Y,L) - \dim_{\mathbf{R}} H^1(Y,L) + \dim_{\mathbf{R}} H^2(Y,L) = 2d(1-g),$$

(similar to the usual Euler-Poincaré formula).

We have isomorphisms $H^i(Y,L) \to H^i(\pi,\rho),\ 0 \le i \le 1$ (since the universal covering of Y, i.e. of X, is a disc we have indeed isomorphisms $H^i(X,L) \to H^i(\pi,\ \rho)$ for all i (cf. §9 Chapter XVI [3]).

Now L is isomorphic to its dual local system since ρ leaves invariant a non-degenerate bilinear form. Therefore by the duality theorem Th. 1.14, [16] we have

$$\dim_{\mathbf{R}} H^0(X,L) = \dim_{\mathbf{R}} H^2(X,L).$$

Thus in this case, we get

$$\dim_{\mathbf{R}} H^1(\pi,\rho) = 2d(g-1) + 2\dim_{\mathbf{R}} H^0(\pi,\rho).$$

In the general case, a similar proof should be possible. However an explicit proof of this proposition along different lines is found in a paper of Weil (§6 and §7, [20]).

Remark 8 Let $U(r)$ (here we change the notation of the Introduction) be the group of unitary matrices of rank r and $\mathcal{U}(r)$ its Lie algebra namely the space of skew Hermitian matrices of rank r (it is a real vector space of dimension r^2.)

If $\theta \in U(r)$, we denote by Ad θ, the adjoint transformation on $\mathcal{U}(r)$, namely if $M \in \mathcal{U}(r)$, $M \mapsto \theta M \theta^{-1}$.

Let $\alpha_1, \ldots, \alpha_k$ be the multiplicities of the distinct roots of the characteristic polynomial of θ i.e. if

$$\theta = A \begin{pmatrix} e^{2\pi i d_1} & & 0 \\ 0 & \ddots & 0 \\ 0 & & e^{2\pi i d_r} \end{pmatrix} A^{-1}, \qquad 0 \leq d_1 \leq \cdots \leq d_r < 1$$

we have $d_1 = \cdots = d_{\alpha_1}, d_{\alpha_1} \neq d_{\alpha_1+1}, d_{\alpha_1+1} = \cdots = d_{\alpha_2}, d_{\alpha_2} \neq d_{\alpha_2+1}, \ldots$ etc.

Then we find

(i) The rank of $(\mathrm{Id} - \mathrm{Ad}\ \theta)$ on $\mathcal{U}(r)$ is

$$r^2 - (\alpha_1^2 + \cdots + \alpha_k^2) = \sum \alpha_i \alpha_j, \text{ since } r = \sum \alpha_i.$$

(For this it suffices to make the computation for a diagonal θ.)

(ii) Make $U(r)$ operate on itself by inner conjugation.
Then the isotropy group at θ for this action is of (real) dimension $\sum \alpha_i^2$ so that the dimension of the orbit through θ is of (real) dimension $r^2 - (\sum \alpha_i^2)$.
Thus we have

(iii) Rank of $(\mathrm{Id} - \mathrm{Ad}\ \theta)$ on $\mathcal{U}(r)$ = the dimension of the orbit through θ for the action of $U(r)$ on itself by inner conjugation.

Let ρ be a representation of π into $U(r)$. Let $\theta_\nu, \rho(C_\nu)$, $1 \leq \nu \leq m$. Let W_ν be the orbits through θ_ν in $U(r)$ for the action of $U(r)$ onto itself by inner conjugation.

Let R be the set all representations ρ of π in $U(r)$ such that $\theta \in W_\nu, 1 \leq \nu \leq m$ (i.e. the *conjugacy classes of $\rho(C_\nu)$ are fixed*). Then we can identify $\rho \in R$ with the point

$$(\rho(A_i),\ \rho(B_i),\ \rho(C_\nu)) \in U(r)^{2g} \times W, \qquad W = \prod_{\nu=1}^{m} W_\nu,$$

$1 \leq i \leq g;\ \ 1 \leq \nu \leq m.$

Let $\chi\colon U(r)^{2g} \times W \to U(r)$ be the real-analytic map defined by

$$(a_1,\ldots,a_g,\ b_1,\ldots,b_g,\ c_1,\ldots,c_m) \mapsto a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_m$$

Then $R = \chi^{-1}(e)$ and R is therefore a closed real-analytic subset of $U(r)^{2g} \times W$.

If ρ is a representation of π into $U(r)$, let us denote by Ad ρ the representation of π into $\mathcal{U}(r)$ defined by

$$((\mathrm{Ad}\rho)(\theta))(M) = \rho(\theta) M \rho(\theta^{-1}), \quad M \in \mathcal{U}(r),\ \theta \in \pi.$$

With the above notations, we have the following

Lemma 3 *Let $\rho \in U(r)^{2g} \times W$. Then the kernel of the differential map $d\chi$ of the map $\chi: U(r)^{2g} \times W \to U(r)$ (defined above) at ρ can be canonically identified with the space $Z^1(\pi,$ Ad $\rho)$ of 1-cocycles of for the action Ad ρ on $\mathcal{U}(r)$.*

Proof. Let $M(r)$ denote the space of all $(r \times r)$ matrices over $\mathbf{C}$.

Let D be the ring of dual numbers over $\mathbf{C}$ i.e. the algebra over $\mathbf{C}$ with basis $1, \varepsilon$ and $\varepsilon^2 = 0$.

If $a \in M(r)$, a tangent vector to the complex manifold $M(r)$ at a can be identified canonically with a *D-valued point* of $M(r)$ i.e. an element of the form $a + \varepsilon a'$. If $a \in GL(r)$, we write this element in the form

$$(a, \alpha) = a + \varepsilon \alpha a, \quad \alpha \in M(r)$$

(to identify a tangent vector (a, α) at a to one at the identity element of $GL(r)$).

If $a \in U(r)$, then a tangent vector (a, α) at a to $M(r)$ is tangent to $U(r)$ if and only if

$$(1) \qquad (a+\varepsilon\alpha a)(a+\varepsilon\alpha a)^* = \mathrm{Id}$$

Where * denotes the conjugate transpose. Now

$$(a+\varepsilon\alpha a)(a+\varepsilon\alpha a)^* = (a+\varepsilon\alpha a)(a^*+\varepsilon a^*\alpha^*) = (aa^*+\varepsilon\alpha aa^*+\varepsilon aa^*\alpha).$$

But $aa^* = \mathrm{Id}$, so that (1) is equivalent to $\alpha+\alpha^* = 0$ i.e. $\alpha \in \mathcal{U}(r)$, the space of skew hermitian matrices.

We call an element $(a,\alpha) = a+\varepsilon\alpha a$ satisfying (1), a *D-valued point of* $U(r)$ and denote by $U(r)(D)$ the set of such points. We note that $U(r)(D)$ is a group under multiplication.

Given $(a,\alpha) \in U(r)(D)$, let us denote by $T_{a,\alpha}$ this affine transformation on the real vector space $\mathcal{U}(r)$ defined by

$$\theta \mapsto T_{a,\alpha}(\theta) = \mathrm{Ad}(a)\theta + \alpha = a\theta a^{-1} + \alpha, \quad \theta \in \mathcal{U}(r).$$

We have

$$\begin{aligned}(a_1,\alpha_1)(a_2,\alpha_2) &= (a_1+\varepsilon\alpha_1 a_1)(a_2+\varepsilon\alpha_2 a_2)\\ &= a_1a_2+\varepsilon[\alpha_1+a_1\alpha_2 a_1^{-1}]a_1a_2\\ &= a_1a_2+\varepsilon[\alpha_1+\mathrm{Ad}(a_1)\alpha_2]a_1a_2.\end{aligned}$$

This gives

$$T_{(a_1,\alpha_1)\circ(a_2,\alpha_2)}(\theta) = \mathrm{Ad}(a_1\cdot a_2)(\theta) + [\alpha_1+\mathrm{Ad}(a_1)\alpha_2].$$

On the other hand, we check that

$$(T_{a_1,\alpha_1}\circ T_{a_2,\alpha_2})(\theta) = \mathrm{Ad}(a_1a_2)\theta + [\alpha_1+\mathrm{Ad}(a_1)\alpha_2].$$

This shows that $(a,\alpha) \mapsto T_{a,\alpha}$ defines a homomorphism of $U(r)(D)$ into the group $\mathrm{Aff}(\mathcal{U}(r))$ of affine transformations of $\mathcal{U}(r)$.

We see that the kernel of this homomorphism reduces to the scalar matrices of $U(r)$ i.e. $(a,\alpha) \in U(r)(D)$ with $\alpha = 0$ and $a = (\text{scalar})\mathrm{Id}$.

Let us recall that if $\varphi\colon \Gamma \to \mathrm{Aut}\,\mathcal{U}(r)$ (group of vector space automorphisms) is a homomorphism where Γ is a group and $z\colon \Gamma \to \mathcal{U}(r)$ a map, then $z \in Z^1(\Gamma,\varphi)$ if and only if the map $\Gamma \to \mathrm{Aff}(\mathcal{U}(r))$ defined by $\gamma \mapsto$ the element of $\mathrm{Aff}(\mathcal{U}(r))$ defined by $\theta \mapsto \varphi(\gamma)\theta + z(\gamma)$, $\theta \in \mathcal{U}(r)$, is a homomorphism.

Let $\rho : \pi \to U(r)$ be a representation as in the lemma. Then ρ is represented by the point

$$(a_1,b_1,\ldots,a_g,b_g,c_1,\ldots,c_m) \in U(r)^{2g} \times W$$

$$a_i = \rho(A_i),\ b_i = \rho(B_i),\ c_\nu = \rho(C_\nu), \qquad 1 \le i \le g,\ 1 \le \nu \le m.$$

Now a tangent vector to $U(r)^{2g} \times W$ at ρ can be represented by

$$t = (a_i + \varepsilon\alpha_i a_i,\ b_i + \varepsilon\beta_i b_i,\ c_\nu + \varepsilon\gamma_\nu c_\nu),$$

where $a_i, b_i, c_\nu \in U(r)$: $\alpha_i, \beta_i, \gamma_\nu \in \mathcal{U}(r)$; $1 \le i \le g$, $1 \le \nu \le m$, and $(c_\nu + \varepsilon\gamma_\nu c_\nu)^{n_\nu} = \mathrm{Id}$, $1 \le \nu \le m$.

Now it is immediate that t is in the kernel of $d\chi$ at the point ρ if and only if

$$\text{(3)} \qquad \prod_{i=1}^{g} (a_i,\ \alpha_i)(b_i,\ \beta_i)(a_i,\ \alpha_i)^{-1}(b_i, \beta_i)^{-1} \prod_{\nu=1}^{m} (c_\nu,\ \gamma_\nu) = \mathrm{Id}$$

$$(c_\nu,\ \gamma_\nu)^{n_\nu} = \mathrm{Id},\ 1 \le \nu \le m.$$

Now (3) holds if and only if

$$\text{(4)} \qquad \prod_{i=1}^{g} T_{a_i,\alpha_i} T_{b_i,\beta_i} T^{-1}_{a_i,\alpha_i} T^{-1}_{b_i,\beta_i} \prod_{\nu=1}^{m} T_{c_\nu}, = \mathrm{Id}$$

$$T^{n_\nu}_{c_\nu,\gamma_\nu} = \mathrm{Id}. \quad 1 \le \nu \le m,$$

since the map $(a, \alpha) \mapsto T_{a,\alpha}$ defines a homomorphism of $U(r)(D)$ into $\mathrm{Aff}(\mathcal{U}(r)$ whose kernel reduces to scalars in $U(r)$.

Let F be the free group on A_i, B_i, C_ν; $1 \le i \le g$, $1 \le \nu \le m$ and N the kernel of the canonical homomorphism $j\colon F \to \pi$.

Let $h\colon F \to \mathrm{Aff}(\mathcal{U}(r))$ be the homomorphism defined by

$$A_i \mapsto T_{a_i,\alpha_i}, \quad B_i \mapsto T_{b_i,\beta_i}, \quad C_\nu \mapsto T_{c_\nu\gamma_\nu}, \qquad 1 \le i \le g,\ 1 \le \nu \le m.$$

Then (4) is equivalent to saying that h is trivial on N i.e. (4) is equivalent to saying that h induces a homomorphism of π into $\mathrm{Aff}(\mathcal{U}(r))$.

By the remark made above relating 1-cocycles with homomorphisms into $\mathrm{Aff}(\mathcal{U}(r))$, we see that (3) is satisfied if and only if there is a 1-cocycle $z\colon \pi \to \mathcal{U}(r)$ in $Z'(\pi, \mathrm{Ad}\,\rho)$ such that

$$z(A_i) = \alpha_i, \quad z(B_i) = \beta_i, \quad z(C_\nu) = \gamma_\nu, \qquad 1 \le i \le g;\ 1 \le \nu \le m.$$

From this the lemma follows immediately since an element $z \in Z^1(\pi, \mathrm{Ad}\,\rho)$ is uniquely determined by its values on $A_i, B_i,\ C_\nu,\ :\ 1 \le i \le g,\ 1 \le \nu \le m$.

We have now

$$\dim_{\mathbf{R}} Z^1(\pi,\ \mathrm{Ad}\,\rho) = \dim_{\mathbf{R}} H^1(\pi,\ \mathrm{Ad}\,\rho) + \text{ space of coboundaries}$$
$$= \dim_{\mathbf{R}} H^1(\pi,\ \mathrm{Ad}\,\rho) + [\dim \mathcal{U}(r) - \dim_{\mathbf{R}} H^0(\pi,\ \mathrm{Ad}\,\rho)].$$

Then by applying Proposition 7, and the Remark following Proposition 7 we get

$$\dim_{\mathbf{R}} Z^1(\pi, \mathrm{Ad}\,\rho) = 2r^2(g-1) + r^2 + \dim_{\mathbf{R}} H^0(\pi, \mathrm{Ad}\,\rho) + \sum_{\nu=1}^{m} \dim W_\nu$$
$$= (2g-1)r^2 + \dim_{\mathbf{R}} H^0(\pi, \mathrm{Ad}\,\rho) + \sum_{\nu=1}^{m} \dim W_\nu$$

Now $H^0(\pi, \mathrm{Ad}\ \rho) = \mathcal{U}(r)^\pi$ (π-invariant elements under the adjoint representation). The scalar matrices are always in $\mathcal{U}(r)^\pi$ so that $\dim_{\mathbf{R}} H^0(\pi, \mathrm{Ad}\,\rho) \geq 1$.

By the semi-continuity theorem on the kernel of the differential map and the implicit function theorem, we conclude that the set of points $\rho \in R$ such that $\dim H^0(\pi, \mathrm{Ad}\,\rho) = 1$, is smooth at these points.

Now $\dim_{\mathbf{R}} H^0(\pi, \mathrm{Ad}\,\rho) = 1 \Leftrightarrow \rho$ is irreducible (because the representation is unitary). Thus we have

Theorem 3 *Let $R \subset U(r)^{2g} \times W$ be identified with the set of unitary representations of π into $U(r)$ such that $\rho(C_\nu)$ varies over a fixed conjugacy class, say $W_\nu \subset U(r)$, and $W = \prod_{\nu=1}^{m} W_\nu$. Then the subset of irreducible unitary representations R_0 of R is open and if it is non-empty it is smooth of real dimension $(2g-1)r^2 + 1 + \sum_\nu \dim W_\nu$.*

Corollary 1 *Let $U(r)$ operate on R by inner conjugation. Then the equivalence classes of irreducible unitary representations corresponds to the quotient space $R_0/U(r)$.*

Now the scalars in $U(r)$ operate trivially and $PU(r)$, the unitary projective group, operates freely on R_0 and therefore $R_0/PU(r) = R_0/U(r)$ has a natural structure of real analytic manifold of (real) dimension $= 2r^2(g-1) + 2 + \sum_\nu e_\nu$.

Corollary 2 *Let E be a π-bundle associated to a unitary representation $\rho\colon \pi \to U(r)$. Then we have*

$$\dim_{\mathbf{R}} H^1(\pi,\ \mathrm{ad}\,\rho) = \dim_{\mathbf{R}} H^1(X,\ \pi,\ E^* \otimes E).$$

i.e.

$$= 2\dim_{\mathbf{C}} H^1(X,\ \pi,\ E^* \otimes E).$$

Proof. From Proposition 7, we get

(i) $\dim_{\mathbf{R}} H^1(\pi,\ \mathrm{ad}\,\rho) = 2r^2(g-1) + 2\dim_{\mathbf{R}} H^0(\pi,\ \mathrm{ad}\,\rho) + \sum_{\nu=1}^m e_\nu$.
Now we have $H^i(X,\pi,E^* \otimes E) = H^i(Y, p_*^\pi(E^* \otimes E))$.
The Riemann–Roch theorem gives

(ii) $\dim_{\mathbf{C}} H^0(Y,\ \ p_*^\pi(E^* \otimes E)) - \dim_{\mathbf{C}} H^1(Y,\ \ p_*^\pi(E^* \otimes E)) = \deg(p_*^\pi(E^* \otimes E)) - r^2(g-1)$.
We have $H^0(Y,\ p_*^\pi(E^* \otimes E) = H^0(X,\ \pi,\ E^* \otimes E) = H^0(\pi,\ \mathrm{ad},\ \rho)$ (by Proposition 1).

Therefore $\dim_{\mathbf{R}} H^0(\pi, \mathrm{ad}\,\rho) = 2\dim_{\mathbf{C}} H^0(X,\pi,E^* \otimes E)$

Thus comparing (i) and (ii), it suffices to show that (cf. definition preceding Proposition 3)

$$\deg p_*^\pi(E^* \otimes E) = -\frac{1}{2}\sum_{\nu=1}^m e_\nu.$$

Choose points $\{x_\nu\},\ 1 \le \nu \le m$, over the points $\{y_\nu\},\ 1 \le \nu \le m$ of Y. Then the isotropy group π_{x_ν} at x_ν for the π-action on X can be identified (not canonically) with the subgroup of π generated by C_ν and thus we can identify C_ν with a generator α of π_x so that $\rho(\alpha) = \rho(C_\nu) = \theta_\nu$.

Choosing ζ an n_νth root of unity as has been done before, we write

$$\rho(\alpha) = \begin{pmatrix} \zeta^{d_1^\nu} & & 0 \\ & \ddots & \\ 0 & & \zeta^{d_r^\nu} \end{pmatrix}, \quad 0 \le d_1^\nu \le \cdots \le d_r^\nu < n_\nu$$

Now the bundle $E^* \otimes E$ is defined locally at the point x by the representation $\mathrm{Ad}\,\rho\colon \pi_x \to GL(r^2)$ defined by

$$\alpha \mapsto \rho(\alpha) \otimes \rho(\alpha)^{-1}$$

Now $\rho(\alpha) \otimes \rho(\alpha)^{-1}$ is a diagonal $(r^2 \times r^2)$-matrix with elements $\zeta^{d_i^\nu - d_j^\nu}, 1 \le i,\ \ j \le r$ and we observe that if $d_i^\nu - d_j^\nu < 0$, we have $\zeta^{d_i^\nu - d_j^\nu} = \zeta^{n_\nu - (d_j^\nu - d_i^\nu)}$.

Thus if we write $\mathrm{Ad}\,\rho(\alpha)$ in the canonical form i.e.

$$\operatorname{Ad}\rho(\alpha) = \begin{pmatrix} \zeta^{\ell_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{\ell_{r^2}} \end{pmatrix}, \qquad 0 \le \ell_1 \le \cdots < \ell_{r^2} < n_\nu$$

we check easily that if $\alpha_1^\nu, \dots, \alpha_k^\nu$ denote the multiplicities of the distinct roots of $\rho(\alpha)$, we have

$$\frac{1}{n^\nu} - \sum_{i=1}^{r^2} \ell_i = \frac{1}{2} \sum_{\substack{1 \le i,j \le k \\ i \ne j}} \alpha_i^\nu \alpha_j^\nu = \frac{1}{2} e_\nu.$$

This shows that $-\deg p_*^\pi(E^* \otimes E) = \frac{1}{2}\sum_{\nu=1}^m e_\nu$ and the corollary is proved.

Proposition 8 *For the group π as above if $g \ge 2$, then there exists an irreducible unitary representation π of arbitrary rank such that $\rho(C_\nu)$ are arbitrary unitary matrices with the condition* $\det \prod_{\nu=1}^m \rho(C_\nu) = 1$ *and $(\rho(C_\nu))^{n_\nu} = 1$ for $1 \le \nu \le m$ (we see that these conditions are necessary).*

Proof. Given r we can always choose two unitary matrices U_1, U_2 of rank r which form an irreducible set.

Let $\theta = \prod_{\nu=1}^m \rho(C_\nu)$. Then we can find two unitary matrices X, Y of rank r such that

$$U_1 U_2 U_1^{-1} U_2^{-1} X Y X^{-1} Y^{-1} = \theta$$

because given a unitary matrix φ of determinant one and rank r, the equation

$$XYX^{-1}Y^{-1} = \varphi$$

is easily seen to be solvable in unitary matrices. Now the representation $\rho\colon \pi \to U(r)$ defined by

$$\rho(A_1) = U_1,\ \rho(B_1) = U_2,\ \rho(A_2) = X,\ \rho(B_2) = Y,\ \rho(A_i) = \rho(B_i) = \mathrm{Id}, \quad 3 \le i \le g$$

is irreducible. This proves the proposition.

Chapter II

Stable bundles and unitary bundles

Notation. In the following (unless otherwise stated) X will mean a smooth projective curve over $\mathbf{C}$ and π a finite group operating faithfully on X.

Let $Y = X/\pi$ and $p : X \to Y$ the canonical morphism. Let g, h denote the genus of X, Y respectively.

Let $\mathcal{O}_Y(1)$ denote an ample invertible sheaf on Y, and $\mathcal{O}_X(1) = p^*(\mathcal{O}_Y(1))$. Then $\mathcal{O}_X(1)$ is an ample invertible π-sheaf on X. We denote by $\mathcal{O}_X(m)$ the invertible sheaf $\mathcal{O}_X(1)^{\otimes m}$ (m-fold tensor product).

Given a coherent sheaf F on X, we denote by $F(m)$ the sheaf $F \otimes \mathcal{O}_X(m)$.

If V is an algebraic vector bundle on X, we denote by $V(m)$, the vector bundle whose sheaf of germs of sections is $\mathcal{V}(m)$, where $\mathcal{V}$ is the sheaf of section of V.

We consider only *algebraic schemes* over $\mathbf{C}$ (i.e. schemes of finite type over $\mathbf{C}$, terminology as in [7]) and by a point of a scheme, we mean always a closed point unless otherwise stated.

Let $p_0 : \tilde{X} \to X$ be a simply-connected covering of X and $p_1 = p \circ p_0$. Then Y is a quotient of a discontinuous group Γ of (faithful) automorphisms of X and there exists a normal subgroup Γ_0 of Γ such that $\pi = \Gamma/\Gamma_0$ and $\Gamma_0 = \pi_1(X)$.

1. Category of semi-stable bundles on X

A vector bundle V on X (assumed always to be algebraic or holomorphic) is said to be *semi-stable* (respectively *stable*) if for all sub-bundles $W (\neq 0)$ of V, we have

$$\mu(W) = \frac{\deg W}{\operatorname{rk}(W)} \leq \frac{\deg V}{\operatorname{rk} V} = \mu(V) (\text{respectively } \frac{\deg W}{\operatorname{rk} W} < \frac{\deg V}{\operatorname{rk} V}$$

for every proper sub-bundle of V.)

(We call $\mu(W)$the *reduced degree of* W).

We see by an immediate application of the Riemann–Roch theorem that V is semi-stable if and only if for every sub-bundle W of V, we have

$$\frac{\chi(W(m))}{rkW} \leq \frac{\chi(V(m))}{rkV}$$

where for example, $\chi(W(m)) = \dim H^0(W(m)) - \dim H^1(W(m))$.

Since $H^1(V(m)) = 0$ for sufficiently large m, this is equivalent to saying

$$\frac{\chi(W(m))}{rkW} \leq \frac{\dim H^0(V(m))}{rkV} \quad \text{for } m \text{ sufficiently large.}$$

Similarly, we can express the condition of stability.

We say that a π-vector bundle V on X is *π-semi-stable* (*or semi-stable π-bundle)* if the underlying vector bundle of V is semi- stable.

A π-vector bundle V on X is said to be *π-stable* if V is π-semi-stable and for every proper π-sub-bundle W of V, we have $\mu(W) < \mu(V)$.

Let $\mathcal{B}$ (respectively $\mathcal{B}_\pi$) denote the category of vector bundles (respectively π-vector bundles) on X.

We denote by $\mathbf{S}$ (respectively $\mathbf{S}_\pi$) the category of semi-stable vector bundles (respectively π-semi-stable vector bundles) on X of degree zero.

We denote by $\mathbf{S}_n$ (respectively $\mathbf{S}_{\pi,n}$) the full sub-category of $\mathbf{S}$ (respectively $\mathbf{S}_\pi$) consisting of vector bundles of rank n.

Proposition 1 *The category $\mathbf{S}$ (respectively $\mathbf{S}_\pi$) is abelian, artinian and noetherian. In particular, every object in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$) has a Jordan–Hölder series and the Jordan–Hölder theorem holds in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$). Further if $\alpha \in \mathrm{Hom}(V,W)$, V,W in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$), then α is of constant rank on the fibres of V.*

Proof. We make use of the following simple lemmas:

Lemma 1 *Let $\alpha : V \to W$ be a morphism in $\mathcal{B}$ (respectively $\mathcal{B}_\pi$). Then α can be decomposed as:*

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r] & V \arrow[r, "j"] & V_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 & W_2 \arrow[l] & W \arrow[l] & W_1 \arrow[l, "i"'] & 0 \arrow[l]
\end{tikzcd}$$

where V_i, W_i are vector bundles (respectively π-vector bundles), the rows are exact sequences of vector bundles (respectively π-vector bundles),

$\alpha = i \circ \beta \circ j$ and β is a generic isomorphism i.e. β induces an isomorphism on a non-empty open subset of X.

Proof. Let V, W be the $\mathcal{O}_X$-coherent modules associated to V, W respectively and $\alpha : V \to W$ the homomorphism associated to α.

We note that $\mathcal{O}_{X,x}$ is a principal ideal ring. Let W_1 be the minimal $\mathcal{O}_X$-submodule of W containing $\alpha(V)$ such that $W_1 \supset \alpha(V)$ and W/W_1 is locally free (W_1 can be defined as the inverse image of the torsion part of $W/\alpha(V)$).

Now α factorises as

$$\begin{array}{ccccccccc} 0 \to & \ker & \to V & \xrightarrow{j} & \operatorname{Im}\alpha & \to 0 \\ & & & & \downarrow_{\beta} & \\ 0 \leftarrow & W/W_1 & \leftarrow W & \xleftarrow{i} & W_1 & \leftarrow 0 \end{array}$$

$\alpha = i \circ \beta \circ j$, with rows exact and all the $\mathcal{O}_X$-modules occurring in these terms being locally free. The diagram replacing the locally free sheaves by the corresponding vector bundles is the required one.

Lemma 2 *Let $\beta : V \to W$ be a homomorphism of vector bundles of rank r and the same degree on X.*

Then if β is a generic isomorphism, it is in fact an isomorphism.

Proof. By hypothesis $\wedge^r \beta : \wedge^r V \to \wedge^r W$ is a *non-zero* homomorphism and it suffices to prove that $\wedge^r \beta$ is an isomorphism.

Thus to prove the lemma, we can suppose that they are line bundles. Then β can be identified with a non-zero element of $\Gamma(X,\ V^* \otimes W)$.

Now $V^* \otimes W$ is a line bundle of degree zero. This implies that $V^* \otimes W$ the trivial line bundle and β is a constant section ($\neq 0$). This shows that β is an isomorphism.

Proof of proposition. Take the decomposition of α as in Lemma 1, Chap. II.

Since V is semi-stable of degree $0, \deg V_1 \leq 0$.

This implies $\deg V_2 \geq 0$.

Now $\beta : V_2 \to W_1$ is a generic isomorphism so that

$$\deg W_1 \geq \deg V_2 \text{ (for if } r = \operatorname{rk} V_2, \wedge^r \beta : \wedge^r V_2 \to \wedge^r W_1$$

is non-zero, and in this case the property is clear).

This implies that $\deg W_1 = 0$ since $W \in \mathbf{S}$(respectively $\mathbf{S}_\pi$). By Lemma 2, Chap. II, it follows that β is an isomorphism. This proves the proposition.

Let $V \in \mathbf{S}$ (respectively $\mathbf{S}_\pi$). Let $V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a Jordan–Hölder series for V.

Then V_i/V_{i-1} is *stable* (respectively π-stable). We denote by $\mathrm{gr}\,V$ (respectively $gr_\pi V$), the associated graded object $V_1 \oplus V_2/V_1 \oplus \cdots \oplus V_n/V_{n-1}$ (determined only up to isomorphism but not as an object in V). We call V_i/V_{i-1} the *stable components* (respectively π-*stable components)* of $\mathrm{gr}\,V$ (respectively $\mathrm{gr}_\pi V$). Note that the stable bundles (respectively π-stable bundles) are the *simple objects* in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$)..

Let $V \in \mathcal{B}_\pi$ (respectively $\mathcal{B}_\pi$) and L a line (respectively π-line) bundle. Then $V \in \mathbf{S}$ (respectively $\mathbf{S}_\pi$) if and only if $V \otimes L \in \mathbf{S}$ (respectively $\mathbf{S}_\pi$).

Proposition 2 *Let $V, W \in \mathbf{S_r}$ (respectively $\mathbf{S}_{\pi,\mathbf{r}}$) with at least one of them being stable (respectively π-stable). Then if $f : V \to W$ is a non-zero morphism in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$), then it is an isomorphism.*

Proof. Suppose that V is stable. Then V is a simple object in $\mathbf{S}$ (respectively $\mathbf{S}_\pi$). This implies that $\ker f = 0$.

Then by Proposition 1, Chap. II, it follows immediately that f is an isomorphism.

Corollary *Let V be a stable (respectively π-stable) bundle. Then $\mathrm{End}\,V = H^0(X, V^* \otimes V)$ (respectively $\mathrm{End}\,V = H^0(X, \pi, E^* \otimes E)$) reduces to scalars.*

Proof. Let $A = \mathrm{End}_\pi V$ (respectively End V). Then A is a finite-dimensional $\mathbf{C}$-algebra.

By Proposition 1, every non-zero element is a unit. This implies that $A = \mathbf{C}$ and the corollary is proved.

Proposition 3 *For the category $\mathbf{S_r}$, there exists an integer m_0 such that for $m \geq m_0$, we have*

(1) $H^1(V(m)) = 0$ for $m \geq m_0$ (by the Riemann–Roch theorem, this implies that $\dim H^0(V(m))$ is independent of $V \in \mathbf{S}_r$ for $m \geq m_0$).

(2) the canonical homomorphism $E \to V(m)$, where E represents the trivial bundle $X \times H^0(V(m))$ is surjective i.e. $V(m)$ is a quotient bundle of E.

Proof. We note that $V \in \mathcal{B}$ is stable if and only if V^* (dual of V) is stable. We shall now prove the proposition more generally for $\mathbf{S}_r^\alpha$, which denotes the category of semi-stable bundles of rank r and reduced degree α.

Let us first prove (1). If V is semi-stable such that $\mu(V) = \alpha$, then $H^0(V) = (0)$ if $\alpha < 0$, for if there exists $s \in H^0(V)$, $s \neq 0$, then s generates a line sub-bundle L of V such that $\deg L = \mu(L) \geq 0$.

The duality theorem gives that for any $V \in \mathcal{B}$,

$$\dim H^1(V(m)) = \dim H^0(V^*(-m) \otimes K).$$

where K is the line bundle associated to the sheaf of differentials on X.

Now $V^*(-m) \otimes K$ semi-stable if V is semi -stable. Thus $H^1(V(m)) = 0$ if $\mu(V^*(-m) \otimes K) < 0$. Now we have

$$\mu(V^*(-m) \otimes K) = -\mu(V) - m \deg \mathcal{O}_X(1) + \mu(K).$$

Now $\mu(V)$ and $\mu(K)$ are fixed. Thus for $m \gg 0$, the right hand side is negative and so assertion (1) follows.

To prove (2), we proceed as follows: Let I_p denote the ideal sheaf associated to a point $P \in X$ and $T_0 = \mathcal{O}_X / I_p$. Then I_p gives a line bundle of degree -1. Because of (1) we can choose m_0 such that for all $m \geq m_0$, $H^1(V(m)) = 0$ and $H^1(V(m) \otimes I_p) = 0$ for all $P \in X$.

Tensoring the exact sequence $0 \to I_p \to \mathcal{O}_X \to T_p \to 0$ by $\mathcal{O}(m)$ (sheaf associated to $V(m)$), we get

$$0 \to \mathcal{V}(m) \otimes I_p \to \mathcal{V}(m) \to \mathcal{V}(m) \otimes T_p \to 0 \text{ exact.}$$

Writing the cohomology exact sequence, we get

$$H^0(\mathcal{V}(m)) \to H^0(\mathcal{V}(m) \otimes T_p) \to 0 \text{ exact for } m \geq m_0.$$

This implies that the fibre of $V(m)$ at every $P \in X$ is generated by $H^0(\mathcal{V}(m))$ and the proposition is proved. [2]

We say that a family of vector bundles $\{V_t\}_{t \in T}$ on X parametrized by an algebraic (respectively analytic) scheme T is *algebraic* (respectively

[2] This proof suggested by M.S. Narasimhan is more direct than the one to be found in §3, [17]

analytic) if there is an algebraic (respectively analytic) vector bundle V on $X \times T$ such that $V|X \times t \approx V_t$.

We say that a sub-category K of $\mathcal{B}$ is *bounded* if there is an *algebraic* family of vector bundles $\{V_t\}_{t \in T}$ parametrized by an algebraic scheme T such that given $V \in \mathbf{K}$ there is a V_t such that $V \approx V_t$.

Proposition 4 *Let* $\mathbf{K}$ *be a category of vector bundles on X of fixed rank and degree, say n and d respectively, such that it satisfies the conditions (1) and (2) of Proposition 3, Chap. II. Then there is a family of vector bundles $\{V_t\}_{t \in T}$ parametrized by an (irreducible) algebraic variety T "containing"* $\mathbf{K}$ *i.e. given $W \in \mathbf{K}$, there exists $t \in T$ such that $W \approx W_t$ (in particular* $\mathbf{K}$ *is bounded).*

Proof. We make use of the following well-known property (due to Serre).

Lemma 3 *Let V be a vector bundle on X of rank n, $n \geq 2$ and such that $H^0(V)$ generates V. Then there exists a trivial sub-bundle I_{n-1} of V of rank $(n-1)$ so that we have*

$$0 \to I_{n-1} \to V \to V/I_{n-1} \to 0 \text{ exact and } V/I_{n-1} = \wedge^n V.$$

Sketch of proof of lemma. For every $P \in X$, let K_p be the kernel of the canonical homomorphism

$$H^0(V) \to \text{ Fibre of } V \text{ at } P.$$

We have to find an $(n-1)$-dimensional linear subspace of $H^0(V)$ such that its intersection with every K_p is the linear sub-space (0).

This is done easily by counting dimensions.

Let us go to the proof of the proposition. Let m be an integer such that $H^1(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$ for all $V \in \mathbf{K}$.

The dimension of $H^0(V(m))$ is the same for every $V \in \mathbf{K}$.

Let E be the trivial vector bundle on X of rank $= \dim H^0(V(m))$. Then by Lemma 3, (Chapter II), we have

$$0 \to I_{n-1} \to V(m) \to L \to 0 \text{ exact } \forall\, V \in \mathbf{K}.$$

or
$$0 \to I_{n-1}(-m) \to V \to L(-m) \to 0 \qquad \text{exact}$$

We have $\deg V = \deg L(-m) + \deg I_{n-1}(-m)$. It follows then that the degree of $L(-m)$ is constant, say d_1, when V varies over $\mathbf{K}$. Thus every element of $\mathbf{K}$ can be represented as an extension of a line bundle of degree d_1 by the fixed vector bundles $I_{n-1}(-m)$.

Let A denote the affine variety on the vector space $H^1(L_1^* \otimes I_{n-1}(-m))$, where L_1 is a fixed line bundle of degree d_1 so that to each element a of A, we get a vector bundle V_a which is an extension of L_1 by $I_{n-1}(-m)$.

We check easily that $\{V_a\}_{a\in A}$ is in fact an *algebraic family* (cf. §3, [12] for details).

We have an algebraic family $\{L_\alpha\}$ of line bundles of degree 0 parametrized by the Jacobian J of X. Then the correspondence

$$(a,\ \alpha) \to V_a \otimes L_\alpha$$

defines an algebraic family of vector bundles on X parametrized by the irreducible variety $A \times J$. This *contains* the category $\mathbf{K}$ by our construction and the proposition is proved.

Remark 1 The converse of the above proposition is true, namely that, if $\mathbf{K}$ is a bounded category of vector bundles on X, then $H^1(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$ for all $V \in \mathbf{K}$.

When $\mathbf{K}$ reduces to one element, these are the well-known theorems of Serre (cf. [14]).

The general case follows from these theorems and the *semi-continuity theorems* applied to an algebraic family containing $\mathbf{K}$ (cf. §7, Chapter III, [7] or §5, Chapter II [10]).

Given a sub-category $\mathbf{K}$ of $\mathcal{B}$ with bounded ranks and degrees, we see that $\mathbf{K}$ is bounded if and only if there is an algebraic family of vector bundles on X containing $\mathbf{K}$.

We say that a *family of π-vector bundles* $\{V_t\}_{t\in T}$ on X parametrized by an algebraic (respectively analytic) scheme T is *algebraic* (respectively analytic), if there exists an algebraic (respectively analytic) π-vector bundle V on $X \times T$ (for the canonical action of π on $X \times T$ extending the action of π on X by taking the trivial action on T) such that $V|X \times t(\approx X) \approx V_t$.

We say that a category $\mathbf{K}$ of π-vector bundles on X is *bounded* if there is an *algebraic* family of π-vector bundles on X containing $\mathbf{K}$.

Proposition 5 *Let E_1, E_2 be two π-vector bundles on X of the same rank (say r), same π-degree and such that they have the same local type τ (or equivalently locally isomorphic, cf. Remark 1, Proposition 2, Chapter I).*

Then there is an algebraic family $\{E_t\}_{t\in T}$ of π-vector bundles parameterized by an (irreducible) algebraic variety T containing E_1 and E_2 i.e. there exist $t_1, t_2 \in T$ with $E_{t_1} \approx E_1,\ E_{t_2} \approx E_2$ (in the sense of π-isomorphism) and such that all $E_t,\ t \in T$, are locally of type τ.

Proof. Let $y_i \in Y,\ i = 1,\dots,m$, be the points of Y over which $p : X \to Y$ is *ramified* and $n_i,\ i = 1,\dots,m$, the orders of the isotropy groups π_{x_i} at $x_i \in X$ such that $p(x_i) = y_i$.

Let z represent a local coordinate (i.e. a generator of the maximal ideal of the *algebraic* local ring $\mathcal{O}_{Y,y}$) at points of Y.

Then we have matrices (see Remark 4, Chapter I and Remark 5, Chapter I)

$$\Delta_i = \begin{pmatrix} d_1^i/n_i & & 0 \\ & \ddots & \\ 0 & & d_r^i/n_i \end{pmatrix}, \quad 0 \le d_1^i \le d_2^i \le \cdots \le d_r^i < n_i,\ d_j^i \text{ integers}$$

such that E_1, E_2 can be defined by divisors $\Phi^1,\ \Phi^2$ of the form $\Phi^k : y \mapsto \Phi_y^k,\ k = 1,2$ such that

$$\Phi_{y_i}^k = z^{\Delta_i}\Theta_{y_i}^k,\ z^{\Delta_i} = \begin{pmatrix} z^{d_1^i/n_i} & & 0 \\ & \ddots & \\ 0 & & z^{d_r^i/n_i} \end{pmatrix}$$

$\Phi_y^k = \Theta_y^k,\ y \ne y_i$ and $\Theta^k : y \mapsto \Theta_y^k$ represents a divisor for $p_*^\pi(E_k)(k = 1,2)$.

We can in fact find Φ^k so that $\Theta_{y_i}^k$ are *regular* maps into $GL(r)$ in some neighbourhood of y_i (for we can find rational section of the principal bundle associated to a vector bundle regular in a neighbourhood of a finite number of points of X).

Let now E be *any* π-vector bundle on X such that $p_*^\pi(E) \approx p_*^\pi(E_1)$. Then E can be defined by a divisor Φ of the form $\Phi : y \mapsto \Phi_y; \Phi_y = \Theta_y^1,\ y \ne y_i$, and

$$\Phi_{y_i} = z^{\Delta_i}\Psi_i,$$

Ψ_i a regular map of a neighbourhood of y_i into $GL(r)$.

Now by the definition of a divisor, the divisor Φ assigns to y_i the coset in $GL(r, A_{y_i}) \backslash GL(r, L_{y_i})$ determined by Φ_{y_i} (A_{y_i} denotes the power series ring $z^{1/n_{y_i}}$ and L_{y_i} is the quotient field of A_{y_i}).

Consider the equation

$$(*) \qquad u \cdot z^{\Delta_i} \Psi_i = z^{\Delta_i} \Psi_i', \;\; \Psi_i' \in GL(r, \mathcal{O}_{Y, y_i}), \quad u \in GL(r,\ A_{y_i}).$$

If $(*)$ holds, we have $u = z^{\Delta_i} \Psi_i' \Psi_i^{-1} z^{-\Delta_i}$. Set $\alpha = (\alpha_{k1}) = \Psi_i' \Psi_i^{-1}$. Then $\alpha \in GL(r,\ \mathcal{O}_{Y, y_i})$.

Now $(*)$ holds if and only if $z^{\Delta_i} \alpha z^{-\Delta_i} \in GL(r,\ A_{y_i})$. We have

$$z^{\Delta_i} \alpha z^{-\Delta_i} = (\alpha_{k\ell} z^{(d_k^i - d_\ell^i)/n_i}).$$

Now we see easily that $z^{\Delta_i} \alpha z^{-\Delta_i} \in GL(r, A_{y_i})$ if and only if

$$\alpha_{k\ell}(y_i) = 0 \text{ whenever } d_k^i - d_\ell^i < 0.$$

Thus $(*)$ holds if and only if the value of the matrix $\alpha = \psi_i' \psi_i^{-1}$ at y_i belongs to the *parabolic subgroup* of $GL(r)$ determined by matrices $P = (p_{k1})$ such that

$$p_{k1} = 0 \text{ whenever } d_k^i - d_1^i < 0.$$

This shows that we can choose $y \mapsto \Phi_y$ so that

$$\Psi_i = K_i \Theta_{y_i}^1 \text{ for some } K_i \in GL(r).$$

i.e. by modifying $\Theta_{y_i}^1$ by a suitable constant matrix we can represent by a divisor any π-vector bundle E such that $p_*^\pi(E) \approx p_*^\pi(E_1)$.

We get thus a similar statement for π-vector bundles E such that $p_*^\pi(E) \approx p_*^\pi(E_2)$.

Let $V_1 = p_*^\pi(E_1)$ and $V_2 = p_*^\pi(E_2)$. Then $V_1,\ V_2$ are vector bundles on Y of rank r and the same degree. Then by Proposition 4, Chap. II, there is an algebraic family $\{V_t\}_{t \in T_0}$ parametrized by a quasi projective algebraic variety T_0 such that there are two points $t_1, t_2 \in T_0$ such that $V_{t_1} \approx V_1$ and $V_{t_2} \approx V_2$.

We shall now show that there is an open neighbourhood T of T_0 containing t_1 and t_2 such that the family$\{V_t\}_{t \in T}$ can be *lifted* to an algebraic family $\{E_t\}_{t \in T}$ of π-vector bundles on X with $E_{t_1} \approx E_1$ and $E_{t_2} \approx E_2$ and E_t is locally of type τ for every $t \in T$.

This would prove the proposition.

Let V be a vector bundle on $Y \times T_0$ which defines the family $\{V_t\}_{t \in T_0}$. We can now find an open neighbourhood T of t_1, t_2 and an open covering $\{U_\alpha\}$ of $Y \times T$ and a meromorphic section Θ of the principal bundle associated to V such that

(i) the restriction of V to U_α is trivial for every α.
(ii) there exists a unique member say U_i of the covering $\{U_\alpha\}$ such that U_i contains $y_i \times T$ and U_α for $\alpha \neq i$ does not intersect $y_i \times T$.
(iii) let U_i' be the open subset in Y defined by $U_i' = \mathrm{pr}_Y(U_i)$.

Then in $p^{-1}(U_i')(p{:}X \to Y)$, the function $w = z^{\frac{1}{n_i}}$ which is a local coordinate at every one of the points $p^{-1}(y_i)$ is non-vanishing except at the points $p^{-1}(y_i)$.
(iv) the section Θ is *regular* at the points $(y_i,\, t)$ $1 \leq i \leq m$ and $\forall_t \in T$.

Choose trivialisations of V over U_α ; then the section is defined by rational maps Θ_α of U_α into $GL(r)$ such that Θ_i is regular at the points $(y_i, t),\ t \in T$.

Now $\Theta_{\alpha\beta} = \Theta_\alpha \Theta_\beta^{-1}$ is a regular map of $U_\alpha \cap U_\beta$ into $GL(r)$ and define transition functions for V.

We observe that the values of Θ_i at (y_i, t_k) $1 \leq i \leq m,\ 1 \leq k \leq 2$ could be *arbitrarily prescribed* (because we can choose a rational section at y_i assuming arbitrary values at y_i).

Let $q{:}X \times T \to Y \times T$ be the map $q = p \times \mathrm{Id}$ and $\{W_\alpha\}$ the covering, $\{W_\alpha\} = q^{-1}(U_\alpha)$. Then W_α is π-invariant.

Let Φ_α be the rational map of W_α into $GL(r)$, defined by $\Phi_\alpha = q^{-1}(\Theta_\alpha)$ if $\alpha \neq i,\ 1 \leq i \leq m$ and Φ_i defined as follows:

$$\begin{pmatrix} w^{d_1^i} & & 0 \\ & \ddots & \\ 0 & & w^{d_r^i} \end{pmatrix} q^{-1}(\Theta_i), \qquad 1 \leq i \leq m,\ (w = z^{\frac{1}{n_i}}).$$

Now if $\Phi_{\alpha\beta} = \Phi_\alpha\ \Phi_\beta^{-1},\ \Phi_{\alpha\beta}$ is a regular map of $W_{\alpha\beta} = W_\alpha \cap W_\beta$ into $GL(r)$ and define a vector bundle E on X.

By the choice of our Φ_α, π operates on E (see the discussion towards the close of §1 in Chap. I.

Now $\Phi_\alpha(t)$ (value of Φ_α at $t \in T$) is a π-divisor on Y such that its representative at y_i is of the form $z^{\Delta_i}\varphi_i$, where φ_i is a regular map in a neighbourhood of y_i into $GL(r)$.

By the preceding discussion, by suitably choosing the values of Θ_i at $(y_i, t_k)(1 \le i \le m,\ 1 \le k \le 2)$, we can suppose that the divisors $\Phi_\alpha(t_1)$ and $\Phi_\alpha(t_2)$ define π-vector bundles which are isomorphic to E_1 and E_2 respectively.

Now the algebraic family $\{E_t\}_{t\in T}$ of π-vector bundles gives a lifting of the family $\{V_t\}_{t\in T}$ with $E_{t_1} \approx E_1$ and $E_{t_2} \approx E_2$ and E_t locally of type τ for all $t \in T$.

This completes the proof of the proposition.

Remark 2 Let $\{E_t\}_{t\in T}$ be an algebraic (respectively analytic) family of π-vector bundles parametrized by an algebraic (respectively analytic) scheme T. Then the equivalence classes of the local representations associated to E_t at the ramification points of $p\colon X \to Y$ are the same for all $t \in T$, provided T is *connected*.

For this it suffices to show that, assuming $\{E_t\}_{t\in T}$ to be analytic, the above conclusion holds in a suitable neighbourhood of every point of T.

Given a point $t \in T$ and $x \in X$ which is a ramification point of $p\colon X \to Y$, we can choose the open neighbourhood U_x of x (invariant by π_x) and an open neighbourhood T_0 of t such that the restriction of the defining bundle E of $\{E_t\}$ to $U_x \times T_0$ is defined by a representation of π_x (see the Remark 1, Chap. I).

The equivalence class represented by this matrix is the local class of representations at x for every $E_t,\ t \in T_0$. From this the required assertion follows.

2. Reduction to constructing orbit spaces under an algebraic group

We say that a coherent sheaf $\mathcal{F}$ on X has Hilbert polynomial P if

$$P(m) = \chi(\mathcal{F}(m)) = \dim H^0(\mathcal{F}(m)) - \dim H^1(\mathcal{F}(m)), \text{ for every integer } m.$$

We say that a coherent sheaf $\mathcal{F}$ on $X \times T, T$ an algebraic prescheme, is *flat* over T if for all $y \in X \times T, y = (x,t)$, the $\mathcal{O}_{X\times T,y}$ module $\mathcal{F}_y$ is flat over $\mathcal{O}_{T,t}$. Let $\mathcal{F}_t$ denote the restriction of $\mathcal{F}$ to $X \times t \approx X, t \in T$. Then if $\mathcal{F}$ is *flat* over T and T is *connected,* the Hilbert polynomial of $\mathcal{F}_t$ is independent of $t \in T$.

Let now $\mathcal{E}$ be a coherent sheaf on X and P a linear polynomial with integral coefficients.

Let Quot: (Algebraic Schemes) $\longrightarrow$ (Sets) be the functor defined by Quot (T) = Set of coherent sheaves F on $X \times T$ such that (i) $\mathcal{F}$ is a quotient of $\mathrm{pr}_X^*(\mathcal{E})(\mathrm{pr}_X\colon X \times T \to X$ projection onto $X)$ (ii) $\mathcal{F}$ is flat over T and (iii) $\mathcal{F}$ has Hilbert polynomial P with respect to T i.e. the Hilbert polynomial of $\mathcal{F}_t$ for every $t \in T$ is P. Then, by Grothendieck, we know that Quot is representable by a *projective* algebraic scheme $Q(\mathcal{E}/P)$ (over $\mathbf{C}$). (Th. 3.2, [6]).

Because of representability, we have a uniquely determined coherent sheaf on $X \times Q(Q = Q(\mathcal{E}/P))$ such that $\mathcal{F}$ is flat over Q, is a quotient of $\mathrm{pr}_X^*(\mathcal{E})$, has Hilbert polynomial P with respect to Q and is *universal* for all the coherent sheaves $\mathrm{Quot}(T)$ on $X \times T$.

Suppose now that $\mathcal{E}$ is a π sheaf. Then we see that π operates on $\mathrm{Quot}(T)$ and this operation is functorial in T. From this, we conclude that π operates canonically as a group of automorphisms of the scheme $Q = Q(\mathcal{E}/P)$.

Let $Q^\pi(\mathcal{E}/P) = Q^\pi$ represent the canonical closed *subscheme of* π-*invariant points* of $Q(\mathcal{E}/P)$ defined as follows: the (geometric) points of Q^π are precisely the points of Q invariant under π, if $q \in Q^\pi$ choose an affine open subset $U = \mathrm{Spec}\, R$ of Q containing q and which is invariant under π; then the ideal which defines the closed subscheme $Q^\pi \cap U$ of U is the ideal of R generated by elements of the form $f - \alpha \cdot f, f \in R$ and $\alpha \in \pi$.

Now we see easily that Q^π represents in fact the functor $T \mapsto (\mathrm{Quot}(T))^\pi$ (subset of π-invariant elements of $\mathrm{Quot}(T)$).

This results easily from the fact that if $Z = \mathrm{Spec}\, B$ is an affine scheme on which a finite group π operates and J is the ideal of B generated by elements of the form $f - \alpha \cdot f, f \in B$ and $\alpha \in \pi$, then a homomorphism of rings $B \to A$ is π-invariant if an only if it is zero on J.

Thus, if Z is the functor

$$Z\colon (\text{Algebraic Preschemes}) \to (\text{Sets})$$

such that Z represents Z and Z^π is the functor $Z^\pi(T) = (Z(T)^\pi(\pi$-invariant subset of $Z(T))$, then we see that $\mathrm{Spec}\,(B/J)$ represents Z. We deduce the above assertion easily from this.

Now if $\mathcal{G} \in (\mathrm{Quot}(T))^\pi$, we see that $\mathcal{G}$ is in fact a π-coherent sheaf and the canonical homomorphism $\mathrm{pr}_X^*(\mathcal{E}) \to \mathcal{G}$ is in fact a π-homomorphism.

Thus Q^π represents the functor $T \to (\mathrm{Quot}(T))^\pi$ = Set of all π-coherent sheaves $\mathcal{G}$ on $X \times T$ which are (i) π-quotients of E (ii) $\mathcal{G}$ is flat over T and has Hilbert polynomial P with respect to T.

We shall suppose hereafter that $\mathcal{E}$ is the sheaf of germs of sections of a π-vector bundle E of rank p such that its underlying vector bundle on X is trivial. We see that identifying E with a fibre over a point of X, the action of π on this vector bundle is given by a representation of π on E. If we refer to E simply as a vector bundle, it is without its π-structure; when we refer to E as a π-bundle it will be done so explicitly. Then $G = \mathrm{Aut}\,E$ (group of automorphisms of E) can be identified with the full linear group $GL(p)$.

Let $H = \mathrm{Aut}_\pi(\mathcal{E}) = \mathrm{Aut}_\pi(E)$($\pi$-automorphisms of E); then H is a direct product of full linear groups; in particular it is *connected and reductive*. We write often $Q(E/P)$ instead of $Q(\mathcal{E}/P)$ (similarly $Q^\pi(E/P)$.

We see that G (respectively H) operates as a group of automorphisms of the scheme $Q(E/P)$ (respectively $Q^\pi(E/P)$)—this again results from the fact that G operates on Quot (T) (respectively (Quot $(T))^\pi$) functorially with respect to T.

Let R_1 be the subset of $Q = Q(E/P)$, consisting of points $q \in Q$ such that $\mathcal{F}_p$ is *locally free* (we recall that $\mathcal{F}$ is the defining quotient coherent sheaf on $X \times Q$ of $\mathrm{pr}_X^*(\mathcal{E})$ and $\mathcal{F}_q$ is the restriction $\mathcal{F}$ to $X \times q \approx X$).

We observe that the rank of $\mathcal{F}_q$ is the same whatever be $q \in R_1$. Let this be r. Let $R_1^\pi = R_1 \cap Q^\pi$. Let R be the subset of R_1 consisting of $q \in R_1$ such that

(i) the canonical linear map $H^0(E) \to H^0(\mathcal{F}_q)$ is an isomorphism and

(ii) $H^1(\mathcal{F}_q) = 0$.

Let $R^\pi = R \cap Q^\pi$. Let $R^\tau \subset R^\pi$ be the subset of R^π consisting of all $q \in Q^\pi$ such that F_q is *locally of a fixed type* τ i.e. the set of all $q \in R$ such that F_q is *locally isomorphic* (i.e. for every $x \in X, \pi_x$-isomorphic in a suitable π_x-invariant neighbourhood of x) to a fixed $F_{q_0}, q_0 \in R$ (see Remark 1, Proposition 2, Chapter I).

If $q \in R_1$ (respectively R_1^π) we denote by F_q the vector bundle (respectively π-vector bundle) associated to $\mathcal{F}_q$.

Proposition 6

(i) R_1 and R (respectively R_1^π, and R^π, R^τ) are G-invariant (respectively H-invariant) open subschemes of Q (respectively Q^π).

(ii) Let $q_1, q_2 \in R$ (respectively R^{π}); then F_{q_1} is isomorphic (respectively π-isomorphic) to F_{q_2} if and only if q_1, q_2 lie in the same orbit under G-invariant (respectively H), $G = \operatorname{Aut} E, H = \operatorname{Aut}_{\pi} E$.

(iii) R (respectively R^{π}) has a local universal property in the following sense.

Let $\{V_t\}_{t\in T}$ be an algebraic or analytic family of vector bundles (respectively π-vector bundles) on X parametrized by an algebraic scheme T such that

(a) the Hilbert polynomial of every V_t is P

(b) $H^1(V_t)) = 0$ for every $t \in T$.

(c) $H^0(V_t)$ generates V_t for every $t \in T$ (respectively the π-vector bundle on X associated to the canonical π-module $H^0(V_t)$ is π-isomorphic to E and $H^0(V_t)$ generates V_t. In fact, it suffices to assume that $H^0(V_t)$ is π-isomorphic to E for one point in each connected component of T).

Then for every $t \in T$, there is an open neighbourhood T_0 of t and a morphism $f{:}\, T_0 \to R$ (respectivelyR^{π}) such that if $q = f(t), F_q$ is isomorphic (respectively π-isomorphic) to V_t.

(iv) R (respectively R^{τ}) is *smooth and irreducible.*
Further R^{π} is smooth, its connected components being of the form R^{τ} for some τ.

(v) (a) If R is *non-empty*, then

$$\dim R = (r^2(g-1)+1) + (\dim G - 1)$$

where r is the integer such that F_q is of rank $r, q \in R$ ($g =$ genus of X).
(b) Let τ associate the representation ρ_ν of $\pi_{x_\nu}, 1 \le \nu \le m$ (cf. Remark 1, Prop. 2, Chap. I)

$$\rho_\nu(\alpha) = \begin{pmatrix} \zeta^{d_1^\nu} & & 0 \\ & \ddots & \\ 0 & & \zeta^{d_r^\nu} \end{pmatrix}, \qquad 0 \le d_1^\nu \le \ldots \le d_r^\nu < n_\nu$$

where x_ν is a point of X chosen over $y_\nu, \{y_\nu\}$ being the ramification points in y of $p{:}\, X \to Y, \alpha$ is a generator of π_{x_ν} and ζ is an n_ν-th root of unity as chosen often in §2, Chap. I.

Let $\alpha_1^\nu, \dots, \alpha_{k_\nu}^\nu$ be the multiplicities of the distinct roots of the characteristic polynomial of $\rho(\alpha))$ and

$$e_\nu = \sum_{\substack{1\le i,j\le r\\ i\ne j}} \alpha_i^\nu \alpha_j^\nu$$

(See Remark following Chapter I and Corollary 2 of Theorem 2, Chapter I).

Then if R^τ is *non-empty,* we have

$$\dim R^\tau = (r^2(h-1)+1)+1) + \frac{1}{2}\sum_{\nu=1}^{m} e_\nu + (\dim H - 1)$$

where $h =$ genus of Y.

Proof. (i) To prove that R_1 and R_1^π are open in Q and Q^π respectively it suffices to show that R_1 is open in Q.

For this, it suffices to show that if U is an affine subset of X and $q \in R_1$, then there is an open subset W of Q containing q such that the restriction of $\mathcal{F}$ (defining quotient sheaf of $\mathrm{pr}_X^*(\mathcal{E})$ on $X\times Q$) to $U\times W$ is locally free.

Let now V be an affine open subset of Q containing $q, V = \mathrm{Spec} A', U\times V = \mathrm{Spec} B'$ and $\varphi'{:}A' \to B'$ the homomorphism defining the canonical projection $U\times V \to V$. Now the restriction of $\mathcal{F}$ to $U\times V$ is defined by a noetherian B' module F' which is A'-flat.

Let $\mathfrak{m}$ be the maximal ideal of A' defining $q, S = A - \mathfrak{m}, A = A'S^{-1}, B = B'S^{-1}, \varphi{:}A \to B$ the homomorphism induced by φ' and F the B-module $F'S^{\cdot} - 1'$.

Now it suffices to prove that F is a free module over B, for then it is easily seen that there is an open subset W of R_1 containing q such that the restriction of F to $U\times W$ is free.

Thus the above assertion is an easy consequence of

Lemma 4 *Let $\varphi{:}A \to B$ be a homomorphism of rings such that A is a local ring with maximal ideal m. Suppose that for every maximal ideal $\mathfrak{n}$ of $B, \varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$ i.e. $\varphi(\mathfrak{m})B$ (which is denoted by $\mathfrak{m}.B$) is contained in the radical of B. Let F be a B-module of finite type such that it is flat over A.*

Then $F/\mathfrak{m}F$ is free over $B/\mathfrak{m}B$, then F is free over B.

Proof of lemma. Let $\bar{F} = F/\mathfrak{m}F$. Let $f_1, \ldots f_r$ be a basis of $\bar{F}$ over B/mB. Let $f_1, \ldots f_r$ be some elements of F which lift $f_1, \ldots f_r$. Let L be a free B-module of rank r and $\theta: L \to F$ the B-homomorphism which takes a given basis of L into $f_1, \ldots f_r$ respectively.

Since the induced homomorphism $\bar{\theta}: \bar{L} \to \bar{F}$, where $L = L/mL$ is surjective it follows that $m.\ (F/\theta(L)) = 0$.

Therefore, by Nakayama, $F/\theta(L) = 0$ i.e. θ is surjective. Let $L_1 = \ker\theta$. Then we have the exact sequence of B-modules

$$0 \to L_1 \to L \to F \to 0.$$

Tensoring by $A/\mathfrak{m}$ and using $\mathrm{Tor}_1^A(F, A_{\mathfrak{m}}) = 0, F$ being A-flat, we get

$$0 \to L_1 \otimes_A A/\mathfrak{m} \to L \otimes_A A/\mathfrak{m} \to F \otimes_A A/\mathfrak{m} \to 0$$

is exact.

Since $L \otimes_A A/\mathfrak{m} \approx L/\mathfrak{m}L \approx F/\mathfrak{m}F \approx F \otimes A/\mathfrak{m}$,we see that $L_1 \otimes_A A/\mathfrak{m} = 0$ i.e. $L_1 = \mathfrak{m}L_1$

Now L is a B-module of finite type and B being noetherian with L of finite type over B, L_1 is also of finite type over B. Therefore, by applying Nakayama once again we deduce that $L_1 = 0$. Thus $\theta: L \to F$ is an isomorphism and the lemma is proved.

As remarked above, it follows now that R_1 and R_1^π are open in Q and Q^π respectively.

We shall now show that R is open in R_1. Let $q_0 \in R$. Then by the semicontinuity theorem, there is a neighbourhood S of q_0 such that $H^1(F_q) = 0$, for all $q \in S$. Further $(\mathrm{pr}_S)_*(\mathcal{F})$ is locally free on S of rank = rank E and the corresponding vector bundle is denoted by $(\mathrm{pr}_S)_*(F)$.

The restriction of the canonical homomorphism $\mathrm{pr}_X^*(E) \to F$ to $X \times S$ gives rise to a homomorphism α of the vector bundle $(\mathrm{pr}_S)_*(\mathrm{pr}^*(E))$ (which is isomorphic to the trivial bundle of rank r on S) into $(\mathrm{pr}_S)_*(F)$ such that α induces an isomorphism of the fibres of these vector bundles at q_0.

Therefore there is a neighbourhood S_1 of $q_0, S_1 \subset S$ such that α restricted to S_1 is an isomorphism. This implies that for every $q \in S_1$, the canonical homomorphism $H^0(E) \to H^0(F_q)$ is an isomorphism. This shows that R is open in Q. That R^π is open in Q^π is an immediate consequence of the fact that R is open in Q.

It follows now that R^τ is open in R^π by the Remark following Proposition 5 (Chapter II).

The fact that R_1 and R (respectively R_1^π and R^π, R^τ) are G-invariant (respectively H-invariant) is immediate.

This completes the proof of (i).

(ii) Let $q_1, q_2 \in R$ (respectively R^π). Suppose that q_1, q_2 lie in the same orbit under G (respectively H). Then by the definition of the action of G (respectively H), it is immediate that $F_{q_1}.F_{q_2}$ are isomorphic (respectively π-isomorphic).

We observe that this is so even if we take $q_1, q_2 \in Q$ (respectively Q^π). Suppose on the other hand F_{q_1} is isomorphic (respectively π-isomorphic) to F_{q_2}. This gives rise to an isomorphism (respectively π-isomorphic) of $H^0(F_{q_1})$ onto $H^0(F_{q_2})$.

But now we can identify $H^0(F_{q_1})$ and $H^0(F_{q_2})$ canonically with $H^0(E)$. This implies immediately that there is an automorphism (respectively π-automorphism) of E which takes the quotient sheaf F_{q_1} to F_{q_2} i.e. q_1, q_2 lie in the same orbit under G (respectively H). This proves (ii).

(iii) Let V be the defining vector bundle (respectively π-vector bundle) of the family $\{V_t\}_{t\in T}$. Consider $\mathcal{W} = (\mathrm{pr}_T)_*(\mathcal{V})$, where $\mathcal{V}$ is the coherent sheaf associated to V. Then by our hypothesis $\mathcal{W}$ is locally free. Let W be the vector bundle associated to $\mathcal{W}$. The fibre at t of W can be identified canonically with $H^0(V_t)$.

By our hypothesis on the Hilbert polynomial of V_t, it follows further that $\dim H^0(V_t) = \dim H^0(E)$.

Given $t \in T$, there is a neighbourhood T_0 of t such that W restricted to T_0 is trivial. Then $(\mathrm{pr}_{T_0})^*(W)$ is isomorphic (respectively π-isomorphic) to $(\mathrm{pr}_X)^*(E)$ and V can be considered as a quotient (respectively π-quotient) bundle of $\mathrm{pr}_X^*(E)$.

By the universal property of $Q(E/P)$ (respectively Q^π) we have a morphism $f: T_0 \to Q$ such that V_t is isomorphic (respectively π-isomorphic) to $F_q(q = f(t)$ and V_t is the coherent sheaf associated to $V_t)$.

By the definition of R (respectively R^π), we have $f(T_0) \subset R$ (respectively R^τ). This proves (iii)

(iv) We shall first show that R (respectively R^τ) is connected. We shall suppose that R (respectively R^π) is *non-empty.* Otherwise there is nothing to prove.

Given q_1, q_2 in R (respectively R^τ), then by Proposition 4 (respectively Proposition 5 of Chapter II), there is an algebraic family $\{V_t\}_{t\in T}$ of vector bundles (respectively π-vector bundles) parametrized by an (irreducible) algebraic variety T and two points $t_1, t_2 \in T$ such that $V_{t_1} \approx F_{q_1}, V_{t_2} \approx F_{q_2}$ V_{t_1}, V_{t_2} being the coherent sheaves associated to V_t, V_{t_2} respectively.

Let T_0 be the subset of T formed by points t such that (a) $H^1(V_t) = 0$ and (b) $H^0(V_t)$ generates V_t. Then by the same type of arguments as in the proof of (i) above T_0 is open. Now T_0 contains t_1 and t_2 and T_0 is irreducible.

By the universal property (iii) above, we have an open covering by non-empty subsets $\{T_i\}$ of T_0 and morphism $f_i : T \to R$ (respectively R^τ) such that if $q = f(t)$, then $V_t \approx F_q$.

We have $T_i \cap T_j \neq \emptyset$ and because of (ii) above, the G-saturations (respectively H-saturations) of all $f_i(T_i)$ and $f_j(T_j)$ have non-empty intersection for every i, j. Now the G-saturations (respectively H-saturations) of all $f_i(T_i)$ contain q_1, q_2 and $f_i(T_i)$ is an irreducible constructible subset of R (respectively R^τ). Now q_1, q_2 being arbitrary, it follows that R (respectively R^τ) is connected. We have seen that R^τ is open in R^π. Thus R^π is a disjoint union of connected open subsets R^τ for distinct τ. Hence R^τ are the connected components of R^π.

To conclude the proof of (iv), it suffices to show that R (respectively R^τ or R^π) is *smooth*. For $q \in R$ (respectively R^π), let H_q be the vector bundle defined by the exact sequence.

$$0 \to H_q \to E \to F_q \to 0$$

(where F_q is the vector bundle associated to $\mathcal{F}_q$). Then we have the exact sequence

$$0 \to F_q^* \to E^* \to H_q^* \to 0$$

where, for example, by F_q^*, we mean the dual vector bundle F_q.

Tensoring this by F_q, we get the exact sequence

$$0 \to F_q^* \otimes F_q \to E^* \otimes F_q \to H_q^* \otimes F_q \to 0$$

Writing the cohomology exact sequence, we get

$$0 \to H^0(F_q^* \otimes F_q) \to H^0(E^* \otimes F_q) \to H^0(H_q^* \otimes F_q) \to H^1(F_q^* \otimes F_q)$$

$$\to H^1(F^* \otimes F_q) \to H^1(H_q^* \otimes F_q) \to H^2(F_q^* \otimes F_q).$$

(respectively the corresponding cohomology sequence in the category of π-sheaves).

We note that $H^2(F_q^*\otimes F_q)$ (respectively $H^2(X,\pi,F_q^*\otimes F_q))=0$ since X is a curve (respectively since $H^2(X,\pi,F_q^*\otimes F_q)=H^2(Y,p_*^\pi(F_q^*\otimes F_q))$ and Y is a curve).

Further $H^1(E^*\otimes F_q)=0$ for $E^*\otimes F_q$ is a direct sum of copies of the same F_q. We have also $H^1(X,\pi,E^*\otimes F_q)=0$ for, by the duality theorem for π-bundles (Theorem 2, Chapter I), we have

$$\dim H^1(X,\pi,E^*\otimes F_q)=\dim H^0(X,\pi,E^*\otimes F_q^*\otimes K)$$

where K is the line bundle corresponding to the sheaf of differentials (it is canonically a π-bundle). Now

$$H^0(X,\pi,E\otimes F_q^*\otimes K)\subset H^0(X,E\otimes F_q^*\otimes K).$$

But by the usual duality theorem

$$\dim H^0(E\otimes F_q^*\otimes K)=\dim H^1(E^*\otimes F_q)=0 \text{ by hypothesis.}$$

Thus we deduce that

$$H^1(H_q^*\otimes F_q)=0(\text{ respectively } H^1(X,\pi,H_q^*\otimes F_q)=0)$$

and that the following sequence is exact

(A)
$$0\to H^0(F_q^*\otimes F_q)\to H^0(E^*\otimes F_q)\to H^0(H_q^*\otimes F_q)\to H^1(F_q^*\otimes F_q)\to 0$$

(respectively the corresponding cohomology sequence in the category of π-sheaves).

Now by the *differential study* of the scheme $Q(E/P)$ as well as Q^π (cf. §5, [6]), we deduce that the local ring of R (respectively R^π) at q is *formally smooth* (over **C**) (cf. §17, Chap. IV, [7]) i.e. if T is any affine scheme (over **C**) and T_0 a closed subscheme defined by an ideal I with $I^2=0$, then if $f_0\colon T_0\to \operatorname{Spec}\mathcal{O}_{R,q}$ (respectively $\mathcal{O}_{R^\pi,q}$) is a morphism, it can be extended to a morphism $f\colon T\to \operatorname{Spec}\mathcal{O}_{R,q}$(respectively $\mathcal{O}_{R^\pi,q}$), since $H^1(H_q^*\otimes F_q)=0$ (respectively $H^1(X,\pi,H_q^*\otimes F_q)=0$; for this case, one has to repeat the arguments of §5, [6]).

It follows now that $\mathcal{O}_{R,q}$ (respectively $\mathcal{O}_{R^\pi,q}$) is smooth (over **C**) (cf. §17, Chapter IV, [7]), so that R (respectively R) is smooth. This completes the proof of (iv).

(v) Again by the differential study of the scheme Q (respectively Q^π) referred to above (§5, [6]) $H^0(H_1^*\otimes F_q)($ respectively $H^0(X,\pi,H_q^*\otimes F_q))$

can be identified canonically with the Zariski tangent space to R (respectively R^τ) at q.

Since R (respectively R^τ) is smooth and irreducible, we have

$$\dim R(\text{ respectively } \dim R^\tau) = \dim H^0(H_q^* \otimes F_q)$$
$$(\text{respectively } H^0(X,\pi,H_q^* \otimes F_q)).$$

Now because of (A) above, we have

$$\dim R = \dim H^0(E^* \otimes F_q) - \chi(F_q^* \otimes F_q) =$$

and

$$\dim R^\tau = \dim H^0(X,\pi,E^* \otimes F_q) - \chi(F_q^* \otimes F_q)).$$

Now E^* is a trivial bundle, so that we have

$$H^0(E^* \otimes F_q) = H^0(E^*) \otimes H^0(F_q) = H^0(E^* \otimes E).$$

It follows also that

$$H^0(X,\pi,E^* \otimes F_q) = H^0(X,\pi,E^* \otimes E).$$

Thus $H^0(E^* \otimes F_q)$ (respectively $H^0(X,\pi,E^* \otimes F_q)$) identifies canonically with $H^0(E^* \otimes E)$ (respectively $H^0(X,\pi,E^* \otimes E)$) i.e. the space of endomorphisms (respectively π endomorphisms) of E.

The algebraic group of automorphisms of E (respectively π-automorphisms of E) is open in $H^0(E^* \otimes E)$ (respectively $H^0(X,\pi,E^* \otimes E)$).

This shows that

$$\dim R = \dim G - \chi(F_q^* \otimes F_q)$$

and

$$\dim R^\tau = \dim H - \chi(X,\pi,F_q^* \otimes F_q).$$

Now $\deg(F_q^* \otimes F_q) = 0$ and $\operatorname{rk}(F_q^* \otimes F_q) = r^2$. Therefore by the Riemann–Roch theorem, we get

$$-\chi(F_q^* \otimes F_q) = r^2(g-1)$$

This proves that $\dim R = (r^2(g-1)+1) + (\dim G - 1)$.

We have $\chi(X,\pi,F_q^* \otimes F_q) = \chi(Y,p_*^\pi(F_q^* \otimes F_q))$. Again by the Riemann–Roch theorem, we have

$$-\chi(Y,P_*^\pi(F_q^* \otimes F_q)) = r^2(h-1) - \deg p_*^\pi(F_q^* \otimes F_q).$$

On the other hand we have

$$-\deg p_*^{\pi}(F_q^* \otimes F_q) = \frac{1}{2}\sum_{\nu=1} e_\nu$$

as has been verified in the proof of Corollary 2, Chapter I.

Thus we have

$$\dim R^\tau = (r^2(h-1)+1) + \frac{1}{2}\sum_{\nu=1} e_\nu + (\dim H - 1)$$

and the proposition is proved.

Remark 3 We have a little more than what is stated in (iii) of the above proposition. We have in fact a neighbourhood T_0 of t and a morphism $f\colon T_0 \to R$ such that the defining bundle on $X \times T_0$ of $\{V_t\}$ is the inverse image by $\mathrm{Id} f\colon X \times T_0 \to X \times R$(respectively $X \times R^\tau$) of the family of vector bundles on $X \times R$ (respectively $X \times R^\tau$) defined by the restriction of F to $X \times R$ (respectively $X \times R^\tau$).

Corollary (to Proposition 6) ***(Local moduli for a vector bundle with trivial automorphism).*** *Let V_0 be a vector bundle (respectively π-vector bundle) on X such that* $\dim_C \mathrm{End}\ V_0$ *(respectively* $\dim_{\mathbf{C}} \mathrm{End}_\pi V_0) = 1$. *Then there is a holomorphic family $\{V_d\}$ parametrized by an analytic manifold D and a point $d_0 \in D$ such that*

(1) $V_{d_0} \approx V_0$ and $V_{d_1} \neq V_{d_2}$ (i.e. not isomorphic) if $d_1 \neq d_2, d_2 \in D$.

(2) given a holomorphic family $\{W_t\}$ of vector bundles parametrized by an analytic space T and a point $t_0 \in T$ such that $W_{t_0} \approx V_0$, there is an open neighbourhood T_0 of t_0 and a (unique) morphism $f\colon T_0 \to D$ such that the family $\{W_t\}_{t\in T_0}$ is the inverse image of the family $\{V_t\}_{t\in D}$ i.e. the defining bundle of $\{W_t\}_{t\in T_0}$ on $X \times T_0$ is the inverse image of the defining bundle of $\{V_d\}_{d\in D}$ by the analytic map $\mathrm{Id} \times f\colon X \times T_0 \to X \times D$.

(3) We have in particular $V_d \approx W_t$ where $d = f(t)$, and

$$\dim D = \dim H^1(V_0^* \otimes V_0)(\text{respectively} H^1(X, \pi, V_0^* \otimes V_0)).$$

Proof. Choose an integer m such that (a) $H^1(V_0(m)) = 0$ and (b) $H^0(V_0(m))$ generates $V_0(m)$. Let E be the trivial vector bundle $X \times H^0(V_0(m))$ (respectively the π-vector bundle associated to the canonical π-module $H^0(V_0(m))$).

Consider the scheme $Q(E/P)$ (respectively $Q^{\pi}(E/P)), P$ being the Hilbert polynomial of $V_0(m)$. Then there is a unique G-orbit K in R (respectively H-orbit K in R^{τ} for a suitable choice of m) such that $F_q \approx \mathcal{V}_0(m) \Leftrightarrow q \in K(\mathcal{V}_0(m)$ sheaf associated to $V_0(m))$.

We have $H \subset G$ and H contains the scalars matrices $\lambda \cdot \mathrm{Id}, \lambda \in \mathbf{C}$ We note that the scalar matrices in G (respectively H) operate trivially on R (respectively R^{τ}).

Denote by PG(respectivelyPH), the quotient group of G (respectively H) modulo the group of scalar matrices. Then the action of G (respectively H) on R (respectively R^{τ}) gives rise to a canonical action of PG (respectively PH) on R (respectively R^{τ}).

By our hypothesis if $q \in K$ then the isotropy group q for the action of PG (respectively PH) reduces to the identity element of PG (respectively PH).

Then there is a neighbourhood U of K (in the topology of analytic spaces) in the manifold R (respectively R^{τ}) such that PG (respectively PH) operates freely on U, the quotient $D = U/G$ (respectively U/H) exists as a manifold and in fact there is a *section* of R (respectively R^{τ}) over D.

Because of this section, we have a holomorphic family of vector bundles $\{V'_d\}_{d \in D}$ parametrized by D and a point $d_0 \in D$ such that $V'_{d_0} \approx F_q \approx V_0(m)$ for $q \in K$ and $V'_d \approx F_q, d$ being such the image of $q \in U$ under the canonical morphism $U \to D$.

Let $\{V_d\}_{d \in D}$ be the holomorphic family $V_d = V'_d(-m)$. We have $V_{d_0} \approx V_0$.

Suppose now that $\{W_t\}_{t \in T}$ is a holomorphic family of vector bundles as in (2) of the above Corollary; W being the defining bundle of $\{W_t\}$ in $X \times T$. Then by the semi-continuity theorem, we can find an integer m and a neighbourhood T_0 of t_0 such that (a) $H^1(W_t(m)) = 0 \forall t \in T_0$ and (b) $H^0(W_t(m))$ generates $W_t(m)$ for all $t \in T_0$.

Then by the property (iii) of the above proposition, we have a morphism f_1 of a neighbourhood of t_0, which we can assume to be T_0 into R (respectively R^{τ}) such that if

$$q = f(t),\ F_q \approx V_t(m) \quad \text{or } F_q(-m) \approx V_t.$$

We can also suppose that $f_1(T_0) \subset U$. Let $f\colon T_0 \to D$ be the morphism obtained by composing f_1 with the canonical morphism $U \to D = U/G$ (respectively U/H).

If $d = f(t)$, we deduce immediately that $W_t \approx V_d$, and in fact that the defining bundle of $\{W_t\}_{t\in T_0}$ is the inverse image by $\mathrm{Id}\times f{:}X\times T_0 \to X\times D$ of the defining bundle of $\{V_d\}_{d\in D}$ (use also the remark on (iii) of the above proposition at the end of its proof).

Finally, we have $\dim D = \dim R - \dim PG = (\text{respectively} \dim R - \dim PH)$. But we have $\dim R = \dim PG + \dim H^1(V_0^* \otimes V_0))$. This proves (3) of the corollary and the corollary is proved.

Remark 4 Let, as usual $\mathbf{S}_r$ (respectively $\mathbf{S}_{\pi,r}$) be the category of semi-stable (respectively π-semi-stable) vector bundles on X of rank r and degree zero (respectively π degree zero $\Leftrightarrow$ deg is zero) and $\mathbf{S}_\tau$ the sub-category of $\mathbf{S}_{\pi,r}$ consisting of all π-semi-stable vector bundles V in $\mathcal{S}_{\pi,r}$ which are locally of a fixed type τ (cf. Remark 1, Proposition 2, Chapter I).

Then since $\mathbf{S}_r$ is *bounded* (Proposition 3, §1), we can find an integer m such that $H^0(V(m))$ generates $V(m)$ and $H^1(V(m)) = 0 \forall V \in \mathbf{S}_r$.

Let P be the Hilbert polynomial of $V(m), V \in \mathbf{S}_r$. Then every $V(m)$ $V(m), V \in \mathbf{S}_r$ (respectively $V \in \mathbf{S}_\tau$) can be considered as a quotient of the trivial vector bundle E of rank = $\dim H^0(V(m))$ (respectively π-quotient of a suitable π-bundle E such that the underlying vector bundle is trivial and of rank = $\dim H^0(V(m)), V \in \mathbf{S}_r$).

Thus we can associate to each $V(m), V \in \mathbf{S}_r$ (respectively $V \in \mathbf{S}_\tau$) canonically a G-orbit (respectively H-orbit) in $R(E/P)$ (respectively $R^\pi(E/P)$).

This reduces the problem of classification (up to isomorphism) of semi-stable vector bundles (respectively π-semi-stable vector bundles) to a problem of constructing orbit spaces of the subset of R (respectively R^π or R^τ) corresponding to semi-stable (respectively π-semi-stable) vector bundles.

Proposition 7 *Let V_0 be a vector bundle (respectively π-vector bundle) associated to an irreducible unitary representation ρ of Γ_0 (respectively Γ) is open.*

Let $\{V_d\}_{d\in D}(V_{d_0} \approx V_0)$ be the analytic family as in Cor. Prop. 6 (Chap. II) above.

Then there is a neighbourhood D_0 of d_0 such that for all $d \in D_0, V_d$ is isomorphic (respectively π-isomorphic) to a vector bundle (respectively π-vector bundle) associated to an irreducible unitary representation of

Γ_0 (respectively Γ, for the notation Γ_0, see at the beginning of Chapter II).

(ii) Let $\{V_t\}_{t\in T}$ be an analytic family vector bundles (respectively π-vector bundles) on X. Then the subset T_0 of T of points t such that V_t is isomorphic (respectively π- isomorphic) to a vector bundle associated to an irreducible unitary representation of Γ_0 (respectively Γ) is open.

Proof. Let $\mathcal{S}$ be the **C**-analytic space of all representations of Γ_0 in $GL(r,\mathbf{C})$ (respectively be the **C** -analytic space of all representations χ of Γ into $GL(r,\mathbf{C})$ in X such that the π-vector bundle on X defined by χ is locally of type τ). Let U denote the subset of $\mathcal{S}$ corresponding to unitary representations.

We see easily that there is an *analytic* family $\{W_s\}$ of vector bundles (respectively π-vector bundles) on X parametrized by $\mathcal{S}$. There is an $s_0 \in U \subset S$ such that $V_{s_0} \approx V_0$.

By the universal property of $\{V_d\}$ of Corollary to Proposition 6 (Chapter II), we have a canonical analytical map of a neighbourhood of s_0 in D and therefore its restriction to U defines a *continuous* map f of a neighbourhood U_0 of s_0 into D such that if $d = f(t)$, then $V_d \approx W_t$.

Since equivalence classes of unitary representation of Γ_0 (respectively Γ) define isomorphic bundles, we can suppose that U_0 is invariant under the canonical action of the unitary group K of rank r defining equivalent representations.

Thus f defines canonically a continuous map $g\colon U_0/K \to D$ which is *injective*. Because §3, Chapter I, we can suppose that U_0 again consists of points defining irreducible unitary representations of Γ_0(respectively Γ) and besides U_0/K is a manifold whose real dimension is equal to $\dim_{\mathbf{R}} H^1(\Gamma_0, \mathrm{ad}\,\rho)$ (respectively $\dim_{\mathbf{R}} H^1(\Gamma, \mathrm{ad}\,\rho)$).

But on the other hand, we have

$$\dim_{\mathbf{R}} H^1(\Gamma_0, \mathrm{ad}\,\rho) = 2\dim_{\mathbf{C}} H^1(V_0^* \otimes V))$$

$$(\text{respectively} \dim_{\mathbf{R}} H^1(\Gamma, \mathrm{ad}\,\rho) = 2\dim_{\mathbf{C}} H^1(X, \pi, V^* \otimes V)).$$

cf. §3, Theorem 3, Chapter I and its corollaries). But we have $\dim D = \dim H^1(V_0^* \otimes V)$ (respectively $H^1(X, \pi, V^* \otimes V)$), dimension being as a complex manifold. Therefore, as *topological* manifolds, we have

$$\dim D = \dim U_0/K.$$

Since g is *injective* we conclude that g is a local *homeomorphism* by Brouwer's theorem. This proves (i) of the proposition 7.

Now (ii) is an immediate consequence of the local universal property of $\{V_d\}$, namely the property (2) of Corollary to Proposition 6 (Chapter II) and Proposition 7 is proved.

3. Some results from Mumford's geometric invariant theory

In this section, we give a rapid survey of some results from Mumford's geometric invariant theory, chapters 0, 1, 2 and §4 of Chapter 4 [9]; and §2 of [18] which we need.

In this section of Chapter, *we do not conform to the notations mentioned at the beginning of this chapter.* We still keep the convention that *we consider only algebraic schemes defined over* **C** and by a point of a scheme we mean a closed point unless otherwise stated.

Let X be an algebraic scheme on which an affine algebraic group G operates.

A morphism $\varphi\colon X \to Y$ of algebraic schemes is said to be a *good quotient* (of X modulo G) if it has the following properties:

(i) φ is *G-invariant* i.e. the following diagram

$$\begin{tikzcd} X \times G \arrow[r, "\chi"] \arrow[d, "\mathrm{pr}_X"'] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

is commutative, where $\chi\colon X \times G \to X$ is the morphism by which the action $\sigma f G$ on X is defined.

(ii) φ is a surjective affine morphism

(iii) $(\varphi_*(0_X))^G = 0_Y$

(iv) if X_1, X_2 are closed G-invariant subsets of X such that $X_1 \cap X_2$ is empty, then $\varphi(X_1), \varphi(X_2)$ are closed and $\varphi(X_1) \cap \varphi(X_2)$ is empty.

We say that $\varphi\colon X \to Y$ is a *good affine quotient* if φ is a good quotient and Y is affine (this implies that X is also affine).

The first three conditions are equivalent to the following: φ is surjective and for every affine open subset U of $Y, \varphi^{-1}(U)$ is affine and

G-invariant, and the coordinate ring of U can be identified with the G-invariant subring of $\varphi^{-1}(U)$.

Some properties of good quotients are collected together in the following.

Proposition 8

(i) *The property of being a good quotient is local with respect to the base scheme i.e. $\varphi: X \to Y$ is a good quotient if and only if there is an open covering U_i of Y such that every $V_i = \varphi^{-1}(U_i)$ is G-invariant and the induced morphism $\varphi_i: V_i \to U_i$ is a good quotient.*

(ii) *A good quotient is also a categorical quotient i.e. if $\varphi: X \to Y$ is a good quotient, then for every G-invariant morphism $\psi: X \to Z$, there is a unique morphism $\nu: Y \to Z$ such that $\nu \circ \varphi = \psi$.*

(iii) *Transitivity properties: Let X be an algebraic scheme on which an affine algebraic group G scheme operates.*

Let N be a normal closed subgroup of G and H the affine algebraic group G/N. Suppose that $\varphi_1: X \to Y$ is a good quotient (respectively good affine quotient) of X modulo N. Then we have following:

(a) *The action of G goes down into an action of H on Y.*

(b) *If $\varphi_2: Y \to Z$ is a good quotient (respectively good affine quotient) of Y modulo H, then $\varphi_2 \circ \varphi_1: X \to Z$ is a good quotient (respectively good affine quotient) of X modulo G.*

$$\begin{tikzcd} X \arrow[rr, "good"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

(c) *If $\varphi: X \to Z$ is a good quotient (respectively good affine quotient) of X modulo G, there is a canonical morphism $\varphi_2: Y \to Z$ such that $\varphi = \varphi_2 \circ \varphi_1$ and φ_2 is a good quotient (respectively good affine quotient) of Y modulo H.*

$$\begin{tikzcd} X \arrow[rr, "\text{good}"] \arrow[dr, "\text{good}"'] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

(d) If $\varphi: X \to Y$ is a good quotient (modulo G) Z a normal algebraic variety on which G operates and $j: Z \to X$ a proper injective G-morphism (in particular a closed immersion which is a G-morphism), then Z has a good quotient modulo G; in fact it can be identified with the normalization of the reduced subvariety $(\varphi \circ j)(Z)$ in a suitable finite extension of the field of rational functions of $(\varphi \circ j)(Z)$.

The proof of this proposition is quite easy and we leave it as an exercise.

The basic existence theorem on good quotients is the following:

Theorem 1 *Let $X = \operatorname{Spec} A$ be an affine algebraic scheme on which a reductive affine algebraic group G operates (note that we have supposed the ground field to be $\mathbf{C}$). Let $Y = \operatorname{Spec} A^G (A^G = G$-invariant subring of $A)$ and $\varphi: X \to Y$ the canonical morphism induced by $A^G \subset A$. Then Y is an affine algebraic scheme and $\varphi: X \to Y$ is a good affine quotient.*

Outline of proof. It is a classical fact that A^G is finitely generated over $\mathbf{C}$. Therefore Y is an affine algebraic scheme.

Let V be a finite-dimensional vector space over C and be a G-module through a homomorphism $\rho: G \to GL(V)$ of algebraic groups.

Then G being reductive, V is a semi-simple G-module and we have a canonical linear projection $V \to V^G$, called the Reynold's operator, which is functorial in V.

Now given $f \in A$, it can be embedded in a finite-dimensional G-submodule V of A (for the G-module structure on V induced by the right or left regular representation). Because of this, we get a canonical linear projection $p: A \to A^G$.

Suppose now that X_1, X_2 are two closed G-invariant subsets of X such that $X_1 \cap X_2$ is empty. Then there exists $f \in A$ such that f is 0 on X_1 and 1 on X_2. Let $g = p(f)$. Then $g \in A^G$ and we see again that g is 0 on X_1 and

1 on X_2 (this results from the functoriality of the Reynold's operator). Thus we can *separate* X_1 and X_2 by a G-invariant function g on X.

This is the crucial property and this implies easily the property (iv) in the definition of good quotients and the proposition follows. For more details cf. §2, Chapter I [9].

Let X be a closed subscheme of the n-dimensional projective space $\mathbf{P}^n$. An action of an affine algebraic group G on X is said to be *linear* if it comes from a rational representation of G in the affine scheme $\mathbf{A}^{n+1}$ of dimension $(n+1)$. This definition means that we have an action of G on $\mathbf{A}^{n+1} = \operatorname{Spec} \mathbf{C}[X_1, \ldots, X_{n+1}]$ given by a rational representation of G on $\mathbf{A}^{n+1}$ and that if $\mathfrak{a}$ is the graded ideal of $\mathbf{C}[X_1, \ldots, X_{n+1}]$ defining X, then $\mathfrak{a}$ is G-invariant.

We have $X = \mathsf{Proj}.R$, where $R = \mathbf{C}[X_1, \ldots, X_{n+1}]$. We denote by $\hat{X}$ the cone over $X(\hat{X} = \operatorname{Spec} R)$ and by (0) - the vertex of the cone $\hat{X}$.

The action of G on X lifts to an action of G on $\hat{X}$ and this action and the canonical action of the multiplicative group $\mathbf{G}_m$ on $\hat{X}$ (by homothecies) commute. We observe that the canonical morphism $p\colon \hat{X} - (0) \to X$ is a principal fibre space with structure group $\mathbf{G}_m$ and that p is a good quotient (modulo $\mathbf{G}_m$).

Suppose then that X is a closed subscheme of $\mathbf{A}^n$ and that we have a linear action of an affine algebraic group G on X. A point $x \in X$ is said to be *semi-stable* if for some $\hat{x} \in \hat{X} - (0)$ over x, the closure (in $\hat{X}$) of the G-orbit through $\hat{x}$ does not pass through (0).

A point $x \in X$ is said to be *stable* (to be more precise, properly stable) if for some $\hat{x} \in \hat{X} - (0)$ over x, the orbit morphism $\psi_{\hat{x}}\colon G \to \hat{X}$ defined by $g \mapsto \hat{x} \circ g$ is proper.

Since the actions of G and $\mathbf{G}_m$ on $\hat{X}$ commute, we see easily that if $x \in X$ is semi-stable, then for *every* $\hat{x} \in \hat{X} - (0)$ over x, the closure (in $\hat{X}$) of the G-orbit through $\hat{x}$ does not pass through (0). A similar property holds for the stable points of X.

We denote by X^{ss} (respectively X^s) the set of semi-stable (respectively stable) points of X.

With these definitions and notations, we have the following.

Theorem 2 *Let X be a closed subscheme of $\mathbf{P}^n$ defined by a graded ideal* $\mathfrak{a}$ of $\mathbf{C}[X_1, \ldots X_{n+1}]$ *so that $X = \mathsf{Proj}.R, R = \mathbf{C}[X_1, \ldots, X_{n+1}]$. Let there be given a linear action of a reductive affine algebraic group G on X. Let $Y = \mathsf{Proj}.R^G$ and $\varphi\colon X \to Y$ the canonical rational morphism defined by the inclusion $R^G \subset R$. Then we have*

(i) *$x \in X^{ss}$ if and only if there is a homogeneous G-invariant element $f \in R(R_+$ being the subring of R generated by homogeneous elements of degree $\geq 1)$ such that $f(x) \neq 0$.*
Note that X_f is a G-invariant affine open subset of X and $x \in X_f \subset X^{ss}$ so that we have, in particular that X^{ss} is open in X and for every $x \in X^{ss}$, there is a G-invariant affine open subset containing x and contained in X^{ss}. Further (i) implies that φ is defined at $x \in X^{ss}$.

(ii) *$\varphi: X^{ss} \to Y$ is a good quotient and Y is a projective algebraic scheme and*

(iii) *X^s is a φ-saturated open subset i.e. there exists an open subset Y^s of Y such that $X^s = \varphi^{-1}(Y^s)$ and $\varphi: X^s \to Y^s$ is a geometric quotient i.e. distinct orbits (under G) of X^s go into distinct points of Y^s.*

Outline of proof. Let $\hat{Y} = \operatorname{Spec} R^G$ and $\hat{\varphi}: \hat{X} \to \hat{Y}$ the canonical morphism induced by the inclusion $R^G \subset R$. Then by Theorem 1, $\hat{\varphi}: \hat{X} \to \hat{Y}$ is a good affine quotient.

From this it follows easily that $x \in X^{ss}$ if and only if there exists an $f \in R^G$ such that $f(0) = 0$ and $f(\hat{x}) \neq 0$ where $\hat{x}$ is some point in $\hat{X} - (0)$ over x.

Now the homogeneous components of f are also G-invariant and $f((0)) = 0$ implies that there is a homogeneous components f_d of f such that f_d is in R_+ and $f_d(x) \neq 0$. Thus we see that $x \in X^{ss}$ if and only if there is a homogeneous f in R_+ such that $f(x) \neq 0$.

Now the canonical morphism $\hat{\varphi}_f: \hat{X}_f \to \hat{Y}_f$ induced by φ is a good affine quotient by Theorem 1 (Chapter II).

Since R^G is finitely generated over $\mathbf{C}$, $Y = \mathsf{Proj}.R^G$ is a projective algebraic scheme.

We see easily that we have a canonical *morphism* $\varphi_f: X_f \to Y_f$ induced by the inclusion $(R_f^G)^0 \subset (R_f)^0$ where $(R_f)^0$ (respectively $(R_f^G)^0$) indicates the homogeneous elements of degree 0 in the localization R_f of R (respectively R_f^G of R^G) with respect to the multiplicative closed subset of R (respectively R^G) formed by powers of f.

By the local nature of good quotients in Proposition 8, Chapter II, it suffices to prove that $\varphi_f: X_f \to Y_f$ is a good quotient.

But now we observe that the coordinate ring of the affine scheme Y_f is $(R_f^G)^0$ which is precisely $(R_f^0)^G$ i.e. it is the G-invariant subring of $(R_f)^0$ which is the coordinate ring of the affine scheme X_f.

Therefore $\varphi_f: X_f \to Y_f$ is a good quotient by Theorem 1, Chap. II. This proves the assertions (i) and (ii).

Let $\hat{X}^s$ denote the set of points $\hat{x} \in \hat{X}$ such that the orbit morphism $\Psi_{\hat{x}}: G \to \hat{X}$ is proper. Then $\hat{X}^s$ is $p^{-1}(X^s)$, where p is the canonical morphism $\hat{X} - (0) \to X$.

Let U be the subset of $\hat{X}$ consisting of the points $\hat{x} \in \hat{X}$ such that $\dim G = \dim \psi_{\hat{x}}(G)$. By an easy application of the dimension theorem, we see that U is open. Further U is obviously G-invariant.

Let $W = \hat{X} - U$. Then $C = \hat{\varphi}(W)$ is a closed subset of $\hat{Y}$ because $\hat{\varphi}: \hat{X} \to \hat{Y}$ is a good quotient. It is easily seen that $\hat{X}^s = \hat{\varphi}^{-1}(\hat{Y} - C)$. This implies that $\hat{X}^s$ is a $\hat{\varphi}$-saturated open subset of $\hat{X}$ and it follows easily that $\hat{\varphi}: X^s \to \hat{Y}^s = \hat{Y} - C$ is a geometric quotient. Now one sees easily that $\hat{\varphi}$ goes down to a geometric quotient $\varphi: X^s \to Y^s$ as required in (iii) and the theorem follows.

Let $H_{p,r}(E)$ denote the Grassmannian of r-dimensional quotient linear spaces of a p-dimensional vector space E (over $\mathbf{C}$). We have a canonical immersion of $H_{p,r}(E)$ into the projective space associated to $\bigwedge^{p-r} E$ and if $X = H^N_{p,r}(E)$ denotes the N-fold product of $H_{p,r}(E)$, we have a canonical projective embedding of X, namely the Segre embedding of X associated to the canonical projective embedding of $H_{p,r}(E)$.

There is a natural action of $GL(E)$ on $H_{p,r}(E)$ and this induces a natural action (diagonal action) of $GL(E)$ on $H^N_{p,r}(E)$. The restriction of this action to the subgroup $G = SL(E)$ of $GL(E)$ is a *linear* action with respect to the canonical embedding of X.

We denote by X^{ss} (respectively X^s) the set of semi-stable (respectively stable) points of X for the action of G with respect to the canonical projective embedding of X.

With the above notation, we have the following important computational result of Mumford:

Theorem 3 *Let $X = H^N_{p,r}(E)$ and X^s, X^{ss} be as above. Then for $x \in X, x = \{E_i\}_{1 \le i \le N}, E_i$ a quotient linear space of dimension r of $E, x \in X^{ss}$ (respectively X^s) if and only if for every linear subspace (respectively proper linear subspace) F of E, if F_i denotes the canonical image of F in E_i, we have*

$$\frac{\frac{1}{N}\sum_{i=1}^N \dim F_i}{r} \ge \frac{\dim F}{p} \text{ (respectively } >).$$

Outline of proof. (a) Let us call a a rational homomorphism of $\mathbf{G}_m$ into G a one parameter subgroup of an algebraic group of G (abbreviated 1-ps of G).

Let there be a given linear action of a *reductive* algebraic group G on a projective scheme $X \subset \mathbf{P}^n$.

Then a basic result states that $x \in X^{ss}$ (respectively X^s) if and only if it is so with respect to the restriction of the action of G to every 1-ps of G (cf. page 53, Chapter 2, [9]). This fact can be expressed in a quantitative manner as follows:

Suppose there is an action of $\mathbf{G}_m$ on the projective space $\mathbf{P}_n$ induced by a linear action of $\mathbf{G}_m$ on the affine space $\mathbf{A}^{n+1}$. With respect to a suitable coordinate system in $\mathbf{A}^{n+1}$, this action is given by $x = (x_0, \ldots, x_n) \mapsto (\alpha^{r_0} x_0, \ldots, \alpha^{r_n} x_n)$ where α is the canonical coordinate of $\mathbf{G}_m$.

One defines for $x \in \mathbf{P}^n, \mu(x) = \max\{-r_i \mid i \text{ such that } x_i^* \neq 0\}$ where $x^* = (X_i^*)$ is a point of $\mathbf{A}^{n+1}$ over $x \in \mathbf{P}^n$.

In this manner, for the action of G on X, we obtain an integer $\mu(x, \lambda)$ for every $x \in X$ and every 1-ps λ of G.

We note that if y is the specialization of $\alpha \cdot x$ as $\alpha \to 0$, $\mu(x, \lambda) = \mu(y, \lambda)$.

The above result can be expressed as follows (cf. §1, Chapter 2, [9]):

$$x \in X^{ss} \text{ (respectively } X^s) \begin{cases} \mu(x,\lambda) \geq 0 \text{ (respectively } > 0) \\ \text{or } \mu(y,\lambda) \geq 0 \text{ respectively } > 0,\, y \text{ being the} \\ \text{specialization of } \alpha \cdot x \text{ as } \alpha \to 0 \end{cases}$$

The proof of (a) is not difficult. By the definition of stable and semi-stable points of $X \subset \mathbf{P}^n$, we are reduced to proving the following: we are given a linear action of G (assumed reductive, affine) on the affine space $\mathbf{A}^{n+1}$.

Then if $x \in \mathbf{A}^{n+1}$ such that the orbit morphism $\psi_{x,G}\colon G \to \mathbf{A}^{n+1}$ with respect to G is *not proper* (respectively the closure of $\psi_{x,G}(G)$ does not pass through the origin in $\mathbf{A}^{n+1}$) then there is a 1-ps λ of G such that the orbit morphism $\Psi_{x,\lambda}\colon \mathbf{G}_m \to A_{n+1}$ with respect to λ is also *not proper* (respectively the closure of $\Psi_{x,\lambda}(\mathbf{G}_m)$ does not pass through the origin in $\mathbf{A}^{n+1}$). The proof of this is given on pages 53–54 of §1, Chapter 2 [9].

(Note, incidentally that for the case when $G = GL(r)$ or $SL(r)$, a theorem of Iwahori which is used in [9] in the proof, is quite easy).

(b) Let $X = H^N_{p,r}(E)$. If $x \in X$, let us write

$$x = (L^1, \dots, L^N)$$

where $L^i, 1 \leq i \leq N$, denotes a k-dimensional hyperplane of the projective space $P(E)$ associated to E (in the usual sense).

If we take a 1-ps λ of $SL(E)$ in the diagonal form, say $\lambda(\alpha) = (\alpha^{r_i}\delta_y), 0 \leq i, j \leq p-1$, and $r_0 > r_1 > \dots > r_{p-1}$, the specialization y of $\alpha \cdot x$ can be computed explicitly and thereby a formula for $\mu(x,\lambda) = \mu(y,\lambda)$ is obtained.

This formula holds even when $r_0 \geq r_1 \geq \dots \geq r_{p-1}$ and being a linear function of $r_i, \mu(x,\lambda)$ is positive for every 1-ps λ in the diagonal torus T of $\mathrm{SL}(E)$ if and only if it is so for the extreme sets of r_i, namely when

$$(p-1-q) = r_0 = \dots = r_q; r_{q+1} = \dots = r_{p-1} = -(q+1)$$

Writing down this condition and writing it in an $SL(E)$ invariant form, we get the theorem. For details, see §4, Chapter 4 [9].

This theorem is quite nontrivial and represents one of the significant computations made in [9].

4. Stable bundles and stable points in $H^N_{p,r}(E)$

We follow now the notations and conventions made at the beginning of this chapter.

Consider the category $\mathbf{S}_r$ of semi-stable vector bundles on X of rank r and degree 0. Then if m is an integer which is sufficiently large, we have $H^1(V(m) = 0$ and $H^0(V(m))$ generates $V(m)$, for every $V \in \mathbf{S}_r$ Proposition 3, Chapter II, *Fix such an integer m*. Let P be the Hilbert polynomial of $V(m), V \in \mathbf{S}_r$ and E the trivial vector bundle on X of rank $= \dim H^0(V(m))$.

Let $Q = Q(E/P), R = R(E/P)$ and $R_1 = R(E/P)$ be as in §2, Chapter II. Let R^{ss} (respectively R^s) denote the subset of $q \in R$ such that the vector bundle associated to $\mathcal{F}_q$ is semi-stable(respectively stable).

Let us denote by F_q the vector bundle associated to the coherent sheaf $\mathcal{F}_q$ and by F the vector bundle on $X \times R_1$ associated to the restriction of the defining sheaf $\mathcal{F}$ of $\{F_q\}$ to $X \times R_1$.

Let $\mathbf{n}$ denote an ordered set of N points $P_1, \dots, P_N$ on the curve X. Let $\chi_i: R_1 \to H_{p,r}(E)$ be the morphism into the Grassmannian of r dimensional quotient linear spaces of the p-dimensional vector space $H^0(E)$

(we set $p = \dim H^0(E)$) which assigns to $q \in R_1$ the fibre at P_i of the vector bundle F_q considered canonically as a quotient linear space of $H^0(E)$ (if we conform to the notations of 3, we should write $H_{p,r}(H^0(E))$ instead of $H_{p,r}(E)$. We use this notation for simplicity).

Let $\chi{:}R_1 \to H^N_{p,r}(E)$ be the morphism defined by $\chi(q) = \{\chi_i(q)\},\ 1 \le i \le N$. Let $G = GL(E)$ be the group of automorphism of the vector bundle E.

Then we have $G = GL(H^0(E))$ and we see that χ commutes with the canonical actions of G on R_1 and $H^N_{p,r}(E)$ respectively (for the latter it is the diagonal action, cf. §3).

We shall now extend the above morphism $\chi{:}R_1 \to H^N_{p,r}(E)$ to a *multi-valued set mapping* of $Q = Q(E/P)$ into $H^N_{p,r}(E)$ and we shall denote this extension by

$$\Phi = \{\Phi\}_i,\ \Phi_i{:}Q_1 \to H_{p,r}(E), \quad 1 \le i \le N.$$

(if one prefers, Φ is a subset of $Q_1 \times H^N_{p,r}(E)$ and Φ_i is a subset of $Q_1 \times H_{p,r}(E), 1 \le i \le N$).

Suppose now that for $q \in Q$, $\mathcal{F}_q$ is *not* locally free i.e. $q \notin R_1$. Then we have $\mathcal{F}_q = \mathcal{V}_q \oplus T_q$, where $\mathcal{V}_q$ is locally free and T_q is a torsion sheaf (because X is a smooth curve).

Suppose that $P_i \notin \operatorname{Supp} T_q$ (support of T_q). We then define $\Phi_i(q) \in H_{p,r}(E)$ as the fibre of the vector bundle V_q associated $\mathcal{V}_q$ at P_i considered canonically as a quotient linear space of E of dimension r.

Suppose that $P_i \in \operatorname{Supp} T_q$; we then define $\Phi_i(q)$ to be *any* point of $H_{p,r}(E)$.

We thus obtain a multi-valued (set) mapping $\Phi_i{:}Q \to H_{p,r}(E)$. It is easy to see that Φ_i is a *morphism* in a neighbourhood of $q \in Q$ if and only if $P_i \in \operatorname{Supp}.T_q$ (for by Lemma 4 of §2 Chap. II), we see that the defining sheaf $\mathcal{F}$ of $\{F_q\}$ is locally free in a neighbourhood of $(P_i \times q)$ in $X \times Q$ which implies easily that Φ_i is a morphism in a neighbourhood of q).

We see immediately that the graph of Φ_i in $Q \times H_{p,r}(E)$ is closed and contains the closure of the graph of $\chi{:}R_1 \to H^N_{p,r}(E)$.

Then with these notations we have the following basic fact which connects stable (respectively semi-stable) bundles with the stable (respectively semi-stable) points in $H^N_{p,r}(E)$ for the canonical action of G.

Proposition 9 *If m and N (N = cardinality of the set* $\mathbf{n}$ *of points $P_1, \ldots, P_N$) are sufficiently large, then for $q \in Q = Q(E/P), \Phi(q)$ is a* *semi-stable (respectively stable) point of $H^N_{p,r}(E)$ for the canonical action of G if and only if $q \in R^{ss}$ (respectively R^s) i.e. the vector bundle F_q is semi-stable (respectively stable). Recall that $G = \mathrm{SL}(E)$.*

The proof of this proposition, though not difficult, requires some careful analysis. To prove the proposition is to prove equivalently the following two assertions; namely

$$\left.\begin{array}{l} q \in R^{ss} \;\Rightarrow \Phi(q) = \chi(q) \in H^N_{p,r}(E)^{ss} \\ q \in R^{ss} \text{ then } \Phi(q) \text{ is in } H^N_{p,r}(E)^s \text{ if and only if } q \in R^s \end{array}\right\} \qquad (A)$$

and

$$\Phi(q) \in H^N_{p,r}(E)^{ss} \Rightarrow q \in R^{ss} \qquad (B)$$

The proof of (A) is to be found in §7, especially Propositions 7.2, 7.3 and Theorem 7.1 [17].

The proof of (B) is given in §3, Lemma 2, [18] and is more delicate than that of (A).

Here, we outline only a proof of (A) but not of (B).

Outline of proof of (A). We claim for sufficiently large m, we have the following:

(a) if $\mathcal{G}$ is any sub-bundle of $V, V \in \mathbf{S}_r$ such that $\deg \mathcal{G} = 0$, then $H^1(\mathcal{G}(m)) = 0$ and $H^0(\mathcal{G}(m))$ generates $\mathcal{G}(m)$. Further, we have

$$\frac{\dim H^0(\mathcal{G}(m))}{\mathrm{rk}\,\mathcal{G}(m)} = \frac{\dim H^0(V(m))}{\mathrm{rk}\,V(m)}$$

(b) let $\mathcal{G}$ be any sub-bundle of $V, V \in \mathbf{S}_r$ such that $\deg \mathcal{G} < 0$ and $H^0(\mathcal{G}(m)$ *generates* $\mathcal{G}(m)$ *generically* (i.e. there is at least one point $x \in X$ such that $H^0(\mathcal{G}(m))$ generates the fibre of $\mathcal{G}(m)$ at x).

For the proof of (a), we observe that if $\mathcal{G}$ satisfies the hypothesis of (a), then $\mathcal{G}$ is semi-stable i.e. $\mathcal{G} \in \mathbf{S}_k, 1 \le k \le r$.

Now for sufficiently large m, we have $H^0(V(m))$ generates $V(m)$ and $H^1(V(m)) = 0$ for every $V \in \mathbf{S}_k, 1 \le k \le r$.

The equality

$$\frac{\dim H^0(\mathcal{G}(m))}{\mathrm{rk}\,\mathcal{G}(m)} = \frac{\dim H^0(V(m))}{\mathrm{rk}\,V(m)}$$

is an immediate consequence of the Riemann–Roch theorem.

For the proof of (b), we proceed as follows. We have the following simple

Lemma 5 *Let V be a vector bundle on X such that $H^0(V)$ generates V generically. Then we have*

$$\dim H^0(V) \geq \deg V + \operatorname{rk} V.$$

The proof of this lemma is quite simple (cf. Lemma 7.2 [17]).

If $\operatorname{rk} V = 1$, the hypothesis implies that V can be defined by an effective divisor D and the above inequality is obtained by induction on $\deg D$. Then the general case is obtained by induction on $\operatorname{rk} V$.

To continue the proof of (b), we observe that if θ is an integer and $\mathcal{B}_1$ is the category formed of vector bundles $\mathcal{G}$ on X such that $\mathcal{G}$ is a sub-bundle of some $V \in \mathbf{S}_r$ and $\deg \mathcal{G} \geq \theta$, then $\mathcal{B}_1$ is bounded.

For proving this, we note that since $\mathbf{S}_r$ is bounded, there is an integer e such that whenever W is an indecomposable vector bundle on X such that $\deg W \geq e$ and $\operatorname{rk} W \leq r$, the only homomorphism of W into any V, $V \in \mathcal{S}_r$ is the zero one (cf. Proposition 11.1, [12]).

This implies that the degrees of the indecomposable components of any $\mathcal{G} \in \mathcal{B}_1$ are both bounded above and below. Then by a theorem of Atiyah (cf. p. 426, Theorem 3, [1]) it follows that $\mathcal{B}_1$ is bounded.

Fix now the integer θ so that whenever $\mathcal{G}$ is a sub-bundle of some $V \in \mathcal{S}_r$ and satisfies the condition $\deg \mathcal{G} < \theta$, we have

$$\frac{\deg \mathcal{G}}{\operatorname{rk} \mathcal{G}} < -g.$$

Let $\mathcal{G}$ be a sub-bundle of some $V \in \mathbf{S}_r$ such that $H^0(\mathcal{G}(m))$ generates $\mathcal{G}(m)$ generically and $\deg \mathcal{G} < \theta$. Then by Lemma 5, we have

$$\dim H^0(\mathcal{G}(m)) \leq \deg \mathcal{G}(m) + \operatorname{rk}(\mathcal{G}(m))$$

so that

$$\frac{\dim H^0(\mathcal{G}(m))}{\operatorname{rk} \mathcal{G}(m)} \leq \frac{\deg \mathcal{G}(m)}{\operatorname{rk} \mathcal{G}(m)} + 1 = \frac{\deg \mathcal{G}}{\operatorname{rk} \mathcal{G}} + 1 + m \deg L$$

where L is the line bundle defined by $\mathcal{O}_X(1)$.

On the other hand since for sufficiently large m, we have $H^1(V(m)) = 0$ for every $V \in \mathbf{S}_r$, by applying the Riemann–Roch theorem we get

$$\frac{\dim H^0(V(m))}{\operatorname{rk} V(m)} = -g + 1 + m \deg L, \quad m \text{ sufficiently large.}$$

Since we have $-g > \deg \mathcal{G}/\operatorname{rk}\mathcal{G}$, (b) is proved in this case.

Suppose now $\deg \mathcal{G} \geq \theta$ and $\mathcal{G}$ is a sub-bundle of some $V \in \mathbf{S}_r$ i.e. $\mathcal{G} \in \mathcal{B}_1$. Then for sufficiently large m, we have also $H^1(\mathcal{G}(m)) = 0$.

Then by Riemann–Roch theorem we have

$$\frac{\dim H^0(\mathcal{G}(m))}{\operatorname{rk}\mathcal{G}(m)} = \frac{\deg \mathcal{G}}{\operatorname{rk}\mathcal{G}} - g + 1 + m \deg L, \qquad \text{for } m \text{ sufficiently large.}$$

We have $\deg \mathcal{G} < 0$ and this implies immediately that

$$\frac{\dim H^0(\mathcal{G}(m))}{\operatorname{rk}\mathcal{G}(m)} < \frac{\dim H^0(V(m))}{\operatorname{rk} V(m)}$$

i.e. (b) is proved.

Choose now an integer m such that the properties (a) and (b) above hold. Let $V \in \mathbf{S}_r$. Then if L is a proper linear subspace of $H^0(V(m))$, we set

$$\rho(L) = \frac{\frac{1}{N}\sum_{i=1}^{N} \dim L_i}{\dim L} - \frac{r}{p}$$

where $p = \dim H^0(V(m)) = \operatorname{rk} E$ and L_i denotes the canonical image of L in the fibre of $V(m)$ at P_i ($P_1, \ldots, P_N$ the ordered set $\mathbf{n}$ of points on X).

To prove (A), we have to show that $\rho(L) \geq 0$; further if V is semi-stable and not stable, there is an L such that $\rho(L) = 0$ and that if V is stable $\rho(L) > 0$.

Let $\mathcal{G}$ be the unique sub-bundle of $V(m)$ generated by L; set

$$\rho_1(L) = \frac{\operatorname{rk}\mathcal{G}}{\dim L} - \frac{r}{p}; \qquad \rho_2(L) = \frac{\operatorname{rk}\mathcal{G}}{\dim H^0(\mathcal{G})} - \frac{r}{p}.$$

Suppose now that

(i)
$$\frac{\deg \mathcal{G}}{\operatorname{rk}\mathcal{G}} = \frac{\deg V(m)}{\operatorname{rk} V(m)}$$

and that $L = H^0(G)$. In this case V is semi-stable but not stable.
Then we have $\rho(L) = \rho_1(L)$ further $\rho_1(L) = 0$. This implies that $\rho(L) = 0$.
Suppose that

(ii)
$$\frac{\deg \mathcal{G}}{\operatorname{rk}\mathcal{G}} = \frac{\deg V(m)}{\operatorname{rk} V(m)} \text{ and } L \neq H^0(G).$$

Then $\rho_1(L) > \rho_2(L)$. But $\rho_2(L) = 0$ by (a).

Therefore we have $\rho_1(L) > 0$. Suppose that

(iii)
$$\frac{\deg \mathcal{G}}{\operatorname{rk} \mathcal{G}} < \frac{\deg V(m)}{\operatorname{rk} V(m)}$$

Then $\rho_1(L) \geq \rho_2(L)$ and $\rho_2(L) > 0$ by (b) above. Therefore again in this case $\rho_1(L) > 0$. Thus when (i) does not hold, we have $\rho_1(L) > 0$. Now we have

$$\rho_1(L) - \rho(L) = \frac{\frac{1}{N}\sum_{i=1}^{N}(\operatorname{rk} \mathcal{G} - \dim L_i)}{\dim L}$$

Let λ be the number of distinct points $x \in X$ such that L does not generate the fibre of G at x. Then we have

$$(*) \qquad \frac{\lambda \cdot \operatorname{rk}(\mathcal{G})}{N \cdot \dim L} \geq (\rho_1(L) - \rho(L)) \geq 0.$$

Now we have the following

Lemma 6 *Let W be a vector bundle on X and M a subspace of $H^0(W)$ such that M generates W generically. Let μ be the number of distinct points $x \in X$ such that M does not generate the fibre of W at x. Then we have $\mu \leq \deg W$.*

The proof of the lemma is quite easy (cf. Lemma 7.1 [17]) and we do not give it here.

Now this lemma and the inequality $(*)$ above it implies

$$\frac{\deg \mathcal{G} \operatorname{rk}(\mathcal{G})}{N \cdot \dim L} \geq (\rho_1(L) - \rho(L)) \geq 0.$$

We have
$$\frac{\deg \mathcal{G}}{\operatorname{rk} \mathcal{G}} \leq \frac{\deg V(m)}{\operatorname{rk} V(m)}$$

Therefore, we have

$$\frac{\deg V(m) \cdot \operatorname{rk}(\mathcal{G})^2}{N \cdot \operatorname{rk} V(m) \cdot \dim L} \geq (\rho_1(L) - \rho(L)) \geq 0.$$

Since $\dim L \geq 1$, we have in fact

$$(**) \qquad \frac{\deg V(m) \cdot \operatorname{rk}(\mathcal{G})^2}{N \cdot \operatorname{rk} V(m)} \geq (\rho_1(L) - \rho(L)) \geq 0.$$

Now if N is sufficiently large, we conclude we have in fact $\rho(L) > 0$ in cases (ii) and (iii) since we had already $\rho_1(L) > 0$.

This proves the assertion (A).

Corollary 1 (of Proposition 9) *Let $\{V_t\}_{t\in T}$ be an algebraic or analytic family of vector unless on X of degree 0. Then the subset T^{ss} (respectively T^s) consisting of points $t \in T$ such that V_t is semi-stable (respectively stable) is open in T. In particular R^{ss} (respectively R^s) is an open subset of R (notations as in the above proposition).*

Proof. Consider the multi-valued set mapping $\Phi\colon Q \to H^N_{p,r}(E)$ for sufficiently large m and N as in Proposition 9, Chapter II. Let $\chi\colon R \to H^N_{p,r}(E)$ denote the morphism induced by Φ. Then

$$R^{ss} \text{ (respectively } R^s) = \chi^{-1}(H^N_{p,r}(E)^{ss}) \text{ (respectively } \chi^{-1}(H^N_{p,r}(E))^s).$$

Since $H^N_{p,r}(E)^{ss}$ (respectively $H^N_{p,r}(E)^s$) is open Theorem 2, Chapter II, we deduce that R^{ss} (respectively R^s) is an open subset of R. Then by the local universal property of R by Proposition 6, Chapter II, it follows immediately that T^{ss} (respectively T^s) is open in T.

Corollary 2 *Let $\chi\colon R^{ss} \to H^N_{p,r}(E)^{ss}$ be the canonical morphism induced by the multi-valued set mapping $\Phi\colon Q \to H^N_{p,r}(E)$ as in the above preposition. Then if m and N are sufficiently large, χ is a proper morphism; in fact we can find an integer m_0 and an ordered set of points $\mathbf{n}$ on X such that for any integer m and an ordered set of points $\mathbf{n}$ with $m \geq m_0$ and $\mathbf{n} \supset \mathbf{n}_0$, if $\chi\colon R^{ss} \to H^N_{p,r}(E)^{ss}$ is the canonical associated morphism, then χ is a closed immersion.*

Proof. We shall prove first that χ is proper when m and N are sufficiently large.

Let us denote by the same letter Φ the graph of the multi-valued set mapping $\Phi\colon Q \to H^N_{p,r}(E)$. Let Γ be the graph of the morphism $\chi\colon R^{ss} \to H^N_{p,r}(E)^{ss}$ and Ψ the closure of Γ in $Q \times H^N_{p,r}(E)$.

We have $\Phi \supset \Psi$ (see the discussion preceding the above proposition).

By the above proposition, we have (for sufficiently large m and N)

$$\Phi \cap (Q \times H^N_{p,r}(E)^{ss}) = \Gamma.$$

It follows then that

$$\Psi \cap (Q \times H^N_{p,r}(E)^{ss}) = \Gamma.$$

Since Ψ is closed in $Q \times H^N_{p,r}(E)$, the above relation implies that Γ, which by the definition is closed in $R^{ss} \times H^N_{p,r}(E)^{ss}$ is, in fact closed in $Q \times H^N_{p,r}(E)^{ss}$.

Since Q is *projective*, in particular proper, over $\mathbf{C}$, the canonical projection of $Q \times H^N_{p,r}(E)^{ss}$ onto $H^N_{p,r}(E)^{ss}$ is proper and this implies that $\chi : R^{ss} \to H^N_{p,r}(E)^{ss}$ is proper for sufficiently large m and N.

Finally to show that χ is in fact a closed immersion for a choice of m and $\mathbf{n}$ as indicated above, it suffices to show that there exists $\mathbf{n}_0$ such that whenever $\mathbf{n} > \mathbf{n}_0$, χ is an *immersion.*

This is quite easy and in fact a consequence of a more general fact. Let us show for example that there exists $\mathbf{n}$ such that whenever $\mathbf{n} \supset \mathbf{n}_0$, χ is injective.

Now $\{F_q\}_{q \in R}$ represents a family of quotient bundles of E such that if $q_1 \neq q_2$, then the canonical maps $E \to F_{q_1}, E \to F_{q_2}$ represent distinct quotient bundles of E.

It follows then easily that given distinct points $q_1, \ldots, q_s$ of R, there exists a point $P \in X$ such that the fibres of $F_{q_1}, \ldots, F_{q_s}$ at the point P considered canonically as quotient spaces of $H^0(E)$ are *distinct.*

From this one deduces easily (by the well-known *diagonal argument*) that there exists an ordered set of points $\mathbf{n}_0$ such that the canonical morphism $\chi_1 : R \to H^N_{p,r}(E)$ induced by Φ (associated to $\mathbf{n}_0$) is injective.

This proves the assertion regarding injectivity.

To show that there exists n_0 such that $\chi : R^{ss} \to H^N_{p,r}(E)$ is an immersion, we have only to show that the differential map $d\chi$ of χ is injective at the tangent space of every one of the points of R^{ss} (note that R and $H^N_{p,r}(E)$ are smooth).

The proof of this is similar and is left to the reader. This completes the proof of the corollary.

5. Proof of the main results

We say that a vector bundle V on X is *unitary* (respectively *irreducible unitary*) if it is associated to a unitary (respectively irreducible unitary) representation of Γ_0.

We say that a π-vector bundle on X is *π-unitary* (respectively *irreducible π-unitary*) if it is associated to a unitary (respectively irreducible unitary) representation of Γ.

(We recall that $p_0 : \tilde{X} \to X$ represents a simply connected covering of $X, \Gamma_0 = \pi_1(X), Y = X/\Gamma$ and $\pi = \Gamma/\Gamma_0$).

We need the following observation in the sequel:

Let $\{E_t\}_{t \in T}$ be an algebraic (or analytic) family of π-vector bundles on X, defined by a vector bundle E on $X \times T$. Then $t \mapsto \dim H^i(X, \pi, E_t)$ is an upper semi-continuous function.

To see this, let $q : X \times T \to Y \times T$ be the morphism $q = p \times id$ and $F = q_*^\pi(E)$. Then we claim that $p_*^\pi(E_t) = F_t$ (which implies that F is a vector bundle). This claim follows from the fact that "taking π-invariants commutes with base change, in particular taking restriction to the π-closed subscheme $X \times t$ of $X \times T$". We have already seen that $H^i(X, \pi, E_t) \cong H^i(Y, F_t)$, so that the required semi-continuity property follows from the usual semi-continuity theorem for the family $\{F_t\}$ of vector bundles on Y, parametrized by T.

Proposition 10

(i) Let V be a unitary (respectively π-unitary vector bundle on X). Then V is semi-stable (respectively π-semi-stable).

(ii) Let V be an irreducible unitary (respectively irreducible π-unitary) vector bundle on X. Then V is stable (respectively π-stable).

(iii) Let $\{V_t\}_{t \in T}$ be an analytic family of vector bundles (respectively π-vector bundles) on X such that V_t is stable (respectively π-stable) of degree zero for every $t \in T$. Then the subset T_0 of T formed by points $t \in T$ such that V_t is unitary (respectively π-unitary) is a closed subset of T (for the usual topology).

Proof. (i) To prove (i) it suffices to show that if V is unitary (the π-unitary case is a consequence of this), then V is semi-stable. (cf. definition of π-semi-stability §1, Chapter II).

Suppose that V is not semi-stable. Then there exists a sub-bundle W of V such that $\deg W > 0$. Let $k = \operatorname{rk} W$. Then $\wedge^k W$ is a line bundle such that $\deg(\wedge^k W) = \deg W > 0$ and $\wedge^k W$ is a line sub-bundle of $\wedge^k V$. Now $\wedge^k V$ is again unitary so that we can suppose without loss of generality that W is a line bundle.

We can find a line bundle W_1 such that $\deg W_1 = \deg W$ and W_1 has a section $s, s \neq 0$, vanishing at least at one point of X (for W_1 defined by a divisor D with support at a unique point of P and multiplicity at $P = (\deg W)$. Then we have $W_1 = W \otimes L$ where L is a line bundle of degree zero.

Therefore L is unitary (we use here the classical theorem that a line bundle on X of degree zero is unitary, in fact this is also an easy part of the proof of the next theorem) so that $V \otimes L$ is also unitary.

Thus again to prove (i), we can suppose without loss of generality that W has a non-zero section s vanishing at least at one point of X.

But V has only *constant* sections (i.e. if E is the representation space of a representation of Γ_0 defining V, V is associated to the canonical Γ_0-bundle $\tilde{X} \times E$ and Γ_0-sections of this bundle on $\tilde{X}$ are constant i.e. given by Γ_0-invariant elements of E) by Proposition 1, §1, Chapter I. This leads to a contradiction. This proves (i).

(ii) Let V be an irreducible unitary (respectively irreducible π-unitary) vector bundle on X. Then by (i) V is semi-stable (respectively π-semi-stable). Suppose that V is not stable (respectively π-stable). Then there is a sub-bundle (respectively π-sub-bundle) W of V such that $\deg W = 0$.

Let $k = \mathrm{rk}\, W$. Then similar to what we did in (i), we can suppose without loss of generality that $\wedge^k W$ is trivial (respectively trivial as a π-bundle), for this we make use of the fact that a π-line bundle L such that $\deg L = 0$, is π-unitary. This will also follow from the next theorem.

Then there is a non-zero section (respectively π-section) s_0 of $\wedge^k W$.

Now s_0 can be identified canonically with a section (respectively π-section) of $\wedge^k V$ and is denoted by s.

We see that $s(x)$ is a decomposable element (or tensor) of the fibre of $\wedge^k V$ at x.

But $\wedge^k V$ being unitary (respectively π-unitary), s is a *constant section* (in the sense explained in (i) above). Hence s identifies with a decomposable tensor s of $\wedge^k E(E$ as in (i) above) which is Γ_0-invariant (respectively, Γ-invariant). Hence we get a subspace F of E of rank k, which is Γ_0-stable (respectively Γ-stable). This contradicts the irreducible nature of E and (ii) follows.

(iii) Let $r = \mathrm{rk}\, V_t$ for every $t \in T$.

Let S be the $\mathbf{C}$-analytic space of all representations of Γ_0 into $GL(r, \mathbf{C})$ (respectively of all representations of Γ into $GL(r, \mathbf{C})$).

Let U denote the subset of S corresponding to unitary representations. We see easily that there is an *analytic* family $\{W_s\}_{s\in S}$ of vector bundles (respectively π-vector bundles) on X parametrized by S. Let K be the subset of $S\times T$ consisting of point (s,t) such that $\mathrm{Hom}\,(W_s,\ V_t)\neq 0$ (respectively $\pi-\mathrm{Hom}\,(W_s,\ V_t)\neq 0$).

By the semi-continuity theorem, K is closed in $S\times T$. Since U is compact, $K_0=\mathrm{pr}_T(K\cap(U\times T))$ is closed in T.

Now $t\in K_0$ if and only there is a unitary (respectively π-unitary) bundle $W_s,\ s\in U$ such that $\mathrm{Hom}\,(W_s,\ V_t)\neq 0$. Since W_s is semi-stable (respectively π-semi-stable) and V_t is stable (respectively π-stable), this is equivalent to saying $W_s\approx V_t$ (Proposition 2, Chapter II). i.e. $K_0=T_0$. This proves the proposition.

Theorem 4

(i) Let V be an irreducible π-unitary bundle on X (X being a smooth complete curve over $\mathbf{C}$ *on which a finite group π operates); let τ be its local type.*
Then every π-stable bundle W on X which is of degree zero and locally of type τ (Remark 1, Proposition 2, Chapter I) or equivalently of degree zero and locally isomorphic to V (i.e. for every $x\in X$, π_x-isomorphic to V in some π_x-invariant neighbourhood of x) is also (irreducible) π-unitary.

(ii) Suppose that $g=$ genus *of $Y\geq 2$, $Y=X \bmod \pi$. Then every π-stable vector bundle on X of degree zero is π-unitary*

Proof. Choose an integer m such that $H^0(W(m))$ generates $W(m)$ and $H^1(W(m))=0$ for every $W\in\mathbf{S}_r$ – the category of semi-stable vector bundles on X of rank r and degree zero , where $r=\mathrm{rk}\,V$.

By Proposition 10, V is π-stable so that we have also that $H^0(V(m))$ generates $V(m)$ and $H^1(V(m))=0$.

Let $R=R(E/P)$, P being the Hilbert polynomial of $V(m)$, (notations being as in Proposition 6, Chap. II) and E the π-vector bundle on X associated to the canonical π-module $H^0(V(m))$.

Then the quotient π-vector bundle $E\to V(m)$ is in R^τ. Now R^τ is *irreducible* and smooth (Proposition 6, Chap II).

Let $R^{\tau,s}$ denote the subset of R^τ of points $q\in R$ such that the π-vector bundle F_q is π-stable. Then $R^{\tau,s}$ is a *Zariski open subset* of R as will be shown in the proof of the next theorem (Theorem 5). Therefore $R^{\tau,s}$ is also irreducible, in particular *connected*.

By Propositions 7 and 10 (Chap. II), the subset U of points $q \in R^{\tau,s}$ such that if $F_q = V_q(m), V_q$ is π-unitary is *open and closed in* $R^{\tau,s}$ for the topology of the underlying analytic spaces. Now U is *non-empty* by hypothesis. Therefore $U = R^{\tau,s}$ so that we conclude that for every $q \in R^{\tau,s}$ if $F_q = V_q(m)$, then V_q is π-unitary.

Since every π-stable vector bundle W of π-degree zero and of local type τ occurs in $R^{\tau,s}$, the part (i) of the theorem follows. Note that by Corollary 1, Proposition 9, Chapter II, R^{ss} and R^s are Zariski open in R and R^s respectively being defined by q in R such that F_q is stable (respectively semi-stable), so that (i) follows.

(ii) If $g \geq 2$, by Proposition 8, Chapter I there is always an irreducible unitary representation ρ of Γ such that $\rho(C_\nu) = \rho_\nu(C_\nu)$. The type of the π-vector bundle defined by ρ is determined by the conjugacy classes of $\rho_\nu(C_\nu)$. Hence there is always a vector bundle as in (i) and (ii) follows from (i).

Theorem 5 *Suppose that g = genus of $Y \geq 2, Y = X \bmod \pi, X$ being a smooth complete curve over C on which a finite group operates.*

Let $\mathbf{S}_\tau$ denote the category of π-semi-stable vector bundles on X of rank r, degree zero and fixed local type τ (for definition of local type see Remark 1, Proposition 2, Chapter I)

Let $\mathcal{S}_\tau$ denote the set of equivalence classes of objects in $\mathbf{S}_\tau$ under the equivalence relation $V_1 \sim V_2, V_1, V_2 \in \mathbf{S}_\tau$ if and only if $\mathrm{gr}_\pi V_1 = \mathrm{gr}_\pi V_2$, (see the definition included before Proposition 2, Chapter II).

Then on $\mathcal{S}_\tau$ there is a unique structure of a normal projective variety denoted again by $\mathcal{S}_\tau$ such that (i) if $\{V_t\}_{t \in T}$ is an algebraic family of π-vector bundles such that $V_t \in \mathbf{S}_\tau$ for every $t \in T$ then the canonical set map $f : T \to S$ defined by $t \underset{S}{\mapsto} \mathrm{gr}_\pi V_t$ is a morphism and (ii) given another structure $\mathcal{S}'$ having the property (i) the canonical set map $\mathcal{S}_\tau \to \mathcal{S}'$ is a morphism.

Further the underlying topological space of $\mathcal{S}_\tau$ coincides canonically with the topological space of equivalence classes of π-unitary vector bundles on X of local type τ.

Proof. With respect to the category $\mathbf{S}_\tau$ choose an integer m so that $H^0(V(m))$ generates $V(m)$ and $H^1(V(m)) = 0$ for every $V \in S_r$. Let

E be the π-vector bundle on X associated to the canonical π-module $H^0(V(m))$, $V \in \mathbf{S}_\tau$. Let $H = \mathrm{Aut}_\pi E$ and $G = \mathrm{Aut} E$ i.e. the group of automorphisms of the underlying trivial vector bundle of E. Let P be the Hilbert polynomial of $V(m), V \in \mathbf{S}_\tau$.

Let $R = R(E/P)$ and $R^\tau(\subset R)$ be the smooth schemes on which G and H respectively operate as in Proposition 6, Chapter II.

Let R^{ss} (respectively $R^{\tau,ss}$) denote the subset of $q \in R$ (respectively R^τ) such that if $F_q = V_q(m)$ then $V_q \in \mathbf{S}_r$ (respectively $\mathbf{S}_\tau$). Recall that R^{ss} (respectively $R^{\tau,ss}$) is G-stable open (respectively H-stable open) in R (respectively R^τ), which is a consequence of Proposition 6, Corollary 1 of Proposition 9 (Chapter II) and the definition of π-semi-stability.

Let $p = \mathrm{rk} E$. Then for a suitable choice of m and an ordered set $\mathbf{n}$ of points $P_1, \ldots, P_N$, the canonical G-morphism

$$\chi: R^{ss} \to H^N_{p,r}(E)^{ss}$$

is a *closed immersion*. (by Corollary 2, Proposition 9, Chapter II).

Now $H^N_{p,r}(E)^{ss}$ has a good quotient modulo G and this is projective (by Theorems 2 and 3 of Chapter II).

Now χ being a closed immersion, R^{ss} has also a good quotient $\theta: R^{ss} \to M$ modulo G and M is projective (by Proposition 8, Chapter II).

We see easily that R^{ss} has also a good quotient modulo H and it is affine over M. This also holds for closed subschemes of R^{ss} stable under H. In particular we see that $R^{\pi,ss} = R^\pi \cap R^{ss}, R^{\tau,ss} = R^\tau \cap R^{ss}$ (recall that R^τ is a connected component of R^π, by Proposition 6, Chapter II) have good quotients modulo H. We see that $R^{ss}, R^{\tau,ss}$ are smooth varieties so that their good quotients are normal varieties. Let N be the good quotient $R^{\tau,ss}$ modulo H and $\theta_\tau: R^{\tau,ss} \to N$ the canonical quotient morphism. We have a canonical morphism $j: N \to M$. We claim that it is an affine morphism. In fact, we have an affine covering $\{V_i\}$ of M such that $W_i = \theta^{-1}(V_i)$ is affine and G-stable. Then $W_i' = W_i \cap R^{\tau,ss}$ is a closed H-stable subscheme of W_i. In particular it is affine and we see that $j^{-1}(V_i)$ is just the good quotient W_i' mod H, which is affine. This shows that j is an affine morphism. Since M is projective it follows that N is quasi-projective.

We shall now show that $R^{\tau,s}$ is Zariski open in $R^{\tau,ss}$ by an argument similar to the one in [12]. As we saw in the previous theorem (Theorem 4, Chapter II) this would imply that every π-stable vector bundle of rank

r and degree zero on X is (irreducible) π-unitary. We shall prove this by induction on the rank r.

For $r = 1$ there is nothing to prove since π-stability just means it is a π-line bundle so that in this case $R^{\tau,s} = R^{\tau}$. The induction hypothesis implies that every π-stable vector bundle that is of degree zero and rank $\leq (r-1)$ is (irreducible) π-unitary. Let $R^* = R^{\tau,ss} - R^{\tau,s}$. we see then that $q \in R^*$ if and only if (a) there is a non-zero π-homomorphism of a π-stable vector bundle W on X of rank $\leq (r-1)$ and degree zero into V_q (recall $F_q = V_q(m)$, $E \to F_q$ the quotient representing $q \in R^{\tau,ss} \subset Q(E/P)$), or equivalently (b) there is a non-zero π-homomorphism of a π-unitary vector bundle W of rank$\leq (r-1)$ into V_q. We see that we have a family of π-vector bundles on X, parametrized by an algebraic scheme T whose points represent the bundles in (a). Then by an argument similar to the proof of (iii) of Proposition 10, Chapter II, it follows that R^* is the image of a Zariski closed set in $X \times T$. This implies that R^* is constructible. A similar argument using (b) shows that R^* is the image of a closed set in $X \times K$, where K is compact (in the usual topology) so that R^* is closed with respect to the usual topology in $R^{\tau,ss}$, so that R^* is Zariski closed in in $R^{\tau,ss}$ (cf. Lemma 12.2, [12]). Hence $R^{\tau,s}$ is Zariski open in $R^{\tau,ss}$. Note that $R^{\tau,s}$ is non-empty.

Let U denote the set of unitary representations of Γ of rank r and of local type τ. As we have often done before, using an analytic family of π-vector bundles containing all π-vector bundles corresponding to U and the local universal property of R^{τ} (cf. Proposition 6, Chapter II) we get a canonical continuous map $\lambda\colon U \to N$.

By the previous theorem (Theorem 4, Chapter II) the image of λ contains $\theta_\tau(R^{\tau,s})$ which is dense in N (by Chevalley's theorem since θ_τ is dominant) and U being compact it follows that λ *is surjective* and N is projective.

Further, if $u_1, u_2 \in U$ and the representations of corresponding to u_1, u_2 respectively are equivalent, then $\lambda(u_1) = \lambda(u_2)$.

Suppose we show that

$$q_1, q_2 \in R^{\tau,ss},\ \mathrm{gr}_\pi F_{q_1} \neq \mathrm{gr}_\pi F_{q_2} \text{ then } \theta_\tau(q_1) \neq \theta_\tau(q_2) \qquad (*)$$

Then it will follow that λ induces a homeomorphism of the topological space of equivalence classes of unitary representations of Γ of rank r and local type τ with N and in fact that $\theta_\tau(q_1) = \theta_\tau(q_2)$ *if and only if* $\mathrm{gr}_\pi F_{q_1} = \mathrm{gr}_\pi F_{q_2}$. We see that $N \simeq \mathcal{S}_\tau$.

The universal properties of the theorem are now immediate consequences of the local *universal property* of R^{τ} given by Proposition 6, Chapter II.

Thus, to conclude the proof of the theorem it suffices to prove $(*)$. To prove $(*)$, we have to show that if $q_1, q_2 \in R^{\tau,ss}$ as in $(*)$, then

$$\overline{O(q_1)} \cap \overline{O(q_2)} \text{ is empty}$$

where $\overline{O(q_1)}, \overline{O(q_2)}$ represent the closures in $R^{\tau,ss}$ of the orbits $O(q_1)$, $O(q_2)$ through q_1, q_2 respectively under H.

Suppose that $q \in \overline{O(q_1)} \cap \overline{O(q_2)}$. Then we can find a smooth curve C and a morphism $\mu : C \to \overline{O(q_1)}$ such that if ξ is the generic point of C then $\mu(\xi) \in O(q_1)$ and there is a (closed) point $\xi_0 \in C$ such that $\mu(\xi_0) = q$.

We can now suppose without loss of generality that $\mathrm{gr}_{\pi} F_y$ is constant for every y of the form $\mu(x), x \in C, x \neq \xi_0$ (by taking C to be a suitable neighbourhood of ξ_0).

Taking the inverse of the family $\{F_q\}$ by the morphism μ, to prove $(*)$, we are easily reduced to prove the following

Lemma 7 *Let $\{V_t\}_{t\in T}$ be an algebraic family of vector bundles (respectively π-vector bundles) on X parametrized by an irreducible smooth curve T, and let $t_0 \in T$. Suppose that (i) $\forall t, V_t$ is semi-stable of degree zero (respectively π-semi-stable of π-degree zero) and (ii) $\mathrm{gr}\, V_t$(respectively $gr_{\pi} V_t$) is constant for $t \in T, t \neq t_0$. Then $\mathrm{gr}\, V_t = \mathrm{gr}\, V_{t_0}$ (respectively $gr_{\pi} V_t = gr_{\pi} V_t$) for any $t \in T$.*

Proof of lemma Let $\mathrm{gr}\, V_t = W_1 \oplus \cdots \oplus W_k$, W_i being stable of degree zero (respectively π-stable of π-degree zero) for $t \in T$, $t \neq t_0$. The proof is by induction on k.

Let $k = 1$. Let $D = \{t \in T \mid \dim \mathrm{Hom}\,(W_1, V_t)$ (respectively $\dim \mathrm{Hom}_{\pi}(W_1, V_t)) > 0\}$. By the semi-continuity theorem, D is closed in T. But $S \supset T_0$, where $T_0 = T - t_0$. Therefore $D = T$. This implies that $W_1 \approx V_t$ (Proposition 2, Chap. II.) This proves the lemma for the case $k = 1$. Let us go to the general case.

Let $D_1 = \{t \in T \mid \dim \mathrm{Hom}\,(W_1, V_t)$ (resp. $\dim \mathrm{Hom}_{\pi}(W_1, V_t) > 0\}$. Then D_1 is closed in T and $\cup_{\ell} D_{\ell} = T$.

This implies that at least one D_{ℓ} is T; say $D_1 = T$, so that we have $\mathrm{Hom}\,(W_1, V_t)$ (respectively $\mathrm{Hom}_{\pi}(W_1, V_t)) \neq 0$ for every $t \in T$.

We can now suppose that $\dim \operatorname{Hom}(W_1, V_t)$ (respectively, $\operatorname{Hom}(W_1, V_t))$ is *constant* for every $t \in U$ where U is a non-empty open subset of T.

Let V be the defining bundle (respectively π-bundle) on $X \times T$ of the family V_t. Then $P = (\mathrm{pr}_T)_*(\operatorname{Hom}((\mathrm{pr}_X)^*(W_1), V)) \neq 0$ where, $\operatorname{Hom}((\mathrm{pr}_X)^*(W_1, V)$ denotes the sheaf of germs of homomorphisms of $(\mathrm{pr}_X)^*(W_1)$ into V.

We can suppose, without loss of generality, that T is an affine neighbourhood of t_0. Then the above implies that there is a homomorphism $f\colon (\mathrm{pr}_X)^*(W_1) \to V$ and we can suppose without loss of generality that the restriction of f to every $X \times t, t \in T$ is *non-zero* (for $P = P_1 \oplus P_2$, where P_1 is locally free and P_2 is a torsion sheaf and we can take a section of P which comes from that of P_1 and does not vanish on the fibre of the vector bundle associated to P_1 at t_0).

This implies that $(\mathrm{pr}_X^*(W_1)$ can be considered as a sub-bundle of V; let V_1 be the quotient bundle of V by this sub-bundle.

Now the family defined by V_1 satisfies the hypothesis of the lemma and $(\mathrm{gr} V_1)_t$ (respectively $(\mathrm{gr}_\pi V_1)_t$) is of length $< k$. Now the induction works and the lemma is proved.

This completes the proof of the theorem.

Remark 5 (i) Taking π to be the trivial group, we get the theorem for semi-stable vector bundles of fixed rank and degree zero on a smooth complete curve of genus ≥ 2, proved in [17].

Note that the above theorem (Theorem 5), because of Remark 6, Proposition 5, Chapter I, includes also the generalizations of the main theorem of [17] to arbitrary degree (by creating more ramification points with special local types).

(ii) The dimension of the variety $\mathcal{S}_\tau$ is

$$r^2(h-1) + 1 + \frac{1}{2} \sum_{\nu=1}^{m} e_\nu$$

where h = genus of Y, $r =$ rank of the vector bundles in question and e_ν the integers determined by the local type τ (cf. (v), Proposition 6, Chapter II and Proposition 7, Chapter I).

(iii) The variety $\mathcal{S}_\tau$ is *smooth* at the points corresponding to irreducible π-unitary vector bundles or equivalently π-stable vector bundles.

This is an immediate consequence of the corollary to Proposition 6, Chapter II.

(iv) Let $\{V_t\}_{t\in T}$ be a family of π-vector bundles of type τ, parametrized by an algebraic scheme T. Let T^s (respectively T^{ss}) be the subset of T defined by $t \in T$ such that V_t is π-stable (respectively π-semi-stable) of type τ. Then T^s (respectively T^{ss}) is Zariski open.

For the case of usual vector bundles and π-semi-stable vector bundles, this follows from Corollary 1, Proposition 9, Chapter II. For the π-stable case, this is a consequence of the fact that $R^{\tau,s}$ is open in R^{τ}, shown in the course of the proof of the above theorem and the universal property of R^{τ} (cf. Proposition 6, Chapter II).

When T is an analytic space, a similar assertion holds, in fact $T - T^s$ and $T - T^{ss}$ are closed analytic subsets of T.

(v) We saw above that the morphism $j: N \to M$ is affine. On ethe other hand N and M are projective varieties. It follows then that j is a finite morphism.

(vi) We see easily that for $q \in R^{\tau,ss}$, $R^{\tau,s}$ if and only if the H-orbit $O(q)$ in $R^{\tau,ss}$ is closed and $\dim O(q) = \dim\ H - 1 = \dim\ PH(=$ H modulo scalars$)$.

Remark 6 The gaps mentioned in the footnote on p. 1 are in the proofs for the projectivity of N as well as for the openness of $R^{\tau,s}$ in $R^{\tau,ss}$.

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